III. INDEPENDENT-PARTICLE MODEL

A. Independent-particle model for fermions

A solvable many-particle problem is obtained by rewriting the hamiltonian

$$\begin{aligned}\hat{H} &= \hat{T} + \hat{V} \\ &= \hat{H}_0 + \hat{H}_1\end{aligned} \tag{137}$$

where

$$\begin{aligned}\hat{H}_0 &= \hat{T} + \hat{U} \\ \hat{H}_1 &= \hat{V} - \hat{U}\end{aligned} \tag{138}$$

and $\hat{U}$ is a suitably chosen one-body operator. When only $\hat{H}_0$ is considered the corresponding many-particle problem can be solved straightforwardly. Problems including an external field $\hat{U}_{ext}$ do not present additional difficulty. There are various situations in which the choice of $\hat{U}$ is very important. In general, it can be used to include the average effect of interactions. If the actual ground state of the system breaks a symmetry which the hamiltonian respects, the choice of $\hat{U}$ is critical. Systems with spontaneous magnetization provide an example. In such cases it can be fruitful to add to the hamiltonian $\hat{H}_0$ a term which includes the symmetry-breaking effect and yields a noninteracting ground state which displays this behavior. Clearly, using such a starting point in a framework based on perturbation theory suggests better convergence properties than starting from a noninteracting state with the "wrong" symmetry. A possible useful feature of $\hat{U}$ is therefore that it can be used to speed up the convergence of the perturbation expansion. In the nuclear case one can use $\hat{U}$ to localize the nucleons in a well so that, as a starting point, one can use the many-particle states which are eigenstates of $\hat{H}_0$. Not doing this results in having to deal with plane wave many-particle states which are the eigenstates of $\hat{T}$.

In general one may assume that it is somewhat straightforward to solve the sp problem

$$\begin{aligned}H_0 \left|\lambda\right\rangle &= (T + U) \left|\lambda\right\rangle \\ &= \epsilon_\lambda \left|\lambda\right\rangle .\end{aligned} \tag{139}$$

One can then rewrite $\hat{H}_0$ in the $\{|\lambda\rangle\}$ basis as follows

$$\begin{aligned}\hat{H}_0 &= \sum_{\alpha\beta} \left\langle\alpha\right| T + U \left|\beta\right\rangle a^\dagger_\alpha a_\beta \\ &= \sum_{\alpha\beta}\sum_{\lambda\lambda'} \langle\alpha|\lambda\rangle \left\langle\lambda\right| T + U \left|\lambda'\right\rangle \langle\lambda'|\beta\rangle a^\dagger_\alpha a_\beta \\ &= \sum_{\alpha\beta}\sum_{\lambda\lambda'} \langle\alpha|\lambda\rangle\ \epsilon_{\lambda'}\ \delta_{\lambda,\lambda'}\ \langle\lambda'|\beta\rangle a^\dagger_\alpha a_\beta \\ &= \sum_{\lambda}\left\{\sum_{\alpha}\langle\alpha|\lambda\rangle a^\dagger_\alpha\right\}\ \epsilon_\lambda\ \left\{\sum_{\beta}\langle\lambda|\beta\rangle a_\beta\right\} \\ &= \sum_\lambda \epsilon_\lambda a^\dagger_\lambda a_\lambda .\end{aligned} \tag{140}$$

The many-particle eigenkets of $\hat{H}_0$ are of the form

$$|\lambda_1\lambda_2...\lambda_N\rangle = a^\dagger_{\lambda_1} a^\dagger_{\lambda_2} ... a^\dagger_{\lambda_N} \left|0\right\rangle \tag{141}$$

with eigenvalue

$$E = \sum_{i=1}^{N} \epsilon_{\lambda_i} . \tag{142}$$

For obvious reasons such states are called independent particle states since the only correlation included is that the particles cannot occupy the same sp state. The lowest energy state is obtained by filling the lowest levels in accord with the Pauli principle and is usually written as

$$|\Phi_0\rangle = \prod_{\lambda_i < F} a^\dagger_{\lambda_i} |0\rangle \tag{143}$$

where F characterizes the energy above which all levels are empty. This state $|\Phi_0\rangle$ is sometimes referred to as the Fermi sea.

B. Independent-particle description of electrons in atoms

In the description of atoms most aspects of the physics can be understood on the basis of the following hamiltonian (see Ref. [28])

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \sum_{i=1}^{N} \frac{-Ze^2}{|\mathbf{r}_i|} + \frac{1}{2}\sum_{i,j}^{N} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|} + V_{mag}. \tag{144}$$

Note that in Re. [28] atomic units are used. The various parts of this hamiltonian for N electrons correspond, in this order, to: the kinetic energy of the electrons, the attraction to the nucleus of charge Z which is assumed to be infinitely heavy, the Coulomb repulsion between the electrons, and finally the magnetic interactions including the spin-own-orbit interaction, the spin-other-orbit interaction and the less important spin-spin and orbit-orbit interactions. The actual form of the magnetic interactions can be obtained systematically for a single-electron system by using a non-relativistic reduction of the Dirac equation. For a many-electron system this procedure cannot be applied and this problem is still being studied. Since these magnetic interactions are rather small perturbations compared to the other three contributions to the hamiltonian, it is conventional to lump them together in an effective one-body spin-orbit interaction which has been shown to simulate most of the magnetic interaction effects:

$$V_{so}^{eff} = \sum_i \zeta_i\, \mathbf{l}_i \cdot \mathbf{s}_i \tag{145}$$

where the sum runs only over electrons in the open shells and ζ describes the strength of this effective sp spin-orbit interaction.

The hamiltonian for the electrons in an atom can be considered theoretically well founded. Nevertheless, it should be clear that non-relativistic calculations must at some stage be complemented by considering relativity explicitly. Sensible atomic many-body calculations can be performed by neglecting the magnetic interactions altogether. A simple starting point for atomic calculations is provided by the choice

$$H_0 = \sum_{i=1}^{N} h_0(i) \tag{146}$$

with

$$h_0(i) = \frac{\mathbf{p}_i^2}{2m} + \frac{-Ze^2}{r_i} + u(\mathbf{r}_i). \tag{147}$$

The auxiliary sp potential u must contain a large portion of the effect of the Coulomb repulsion between the electrons. The atomic shell model and the types of sp levels that one encounters are already generated by considering the Hydrogen-like hamiltonian in which u is absent but a nuclear charge of Z is maintained. Clearly, the previous results for the lowest energy, non-interacting many-electron state of this H_0 problem can be immediately applied: electrons will occupy the lowest sp energy states in accordance with the Pauli principle, *i.e.* for each energy level there is $(2l+1) * (2s+1)$-fold degeneracy resulting from the rotational invariance and spin-independence of the hamiltonian. An additional, accidental degeneracy exists for this problem which results in sp energies which are only determined by the radial quantum number n.

This degeneracy is lifted when the effect of the closed shells is approximately included in u. A simple example, including u, is provided by considering the alkali atoms. These atoms have one electron outside a closed shell. At large distances one expects the presence of the closed electron shell(s) to screen the nuclear charge leading to an attraction of only one unit of charge. Very close to the nucleus, the electron will experience the full attraction of the nuclear charge. The presence of closed shells is expected to generate a spherically symmetric field. A smooth interpolation between the two extreme cases is therefore a reasonable picture of u. The effect of this effective potential is to lift the accidental degeneracy of the Hydrogen-like hamiltonian in such a way that the level with the highest probability to be close to the nucleus will profit most in energy. This results in a lowering of the s state with respect to the p state and so on, for a given shell with radial quantum number n. This is for example the case for the Sodium atom for which the $1s, 2s$, and $2p$ shell are filled (notation: $(1s)^2(2s)^2(2p)^6$). Suppose the states $|nlm_lm_s\rangle$ are eigenkets of h_0

$$h_0 |nlm_lm_s\rangle = \epsilon_{nl} |nlm_lm_s\rangle . \tag{148}$$

The state representing the ground state, *i.e.* state with the lowest energy, representing Na in this approximation is given by

$$\left| 300m_s, 211\frac{1}{2}, 211-\frac{1}{2}, ..., 100\frac{1}{2}, 100-\frac{1}{2}\right\rangle \tag{149}$$

where for each occupied state the four quantum numbers nlm_lm_s are given. This state can can also be written as

$$a^\dagger_{300m_s} a^\dagger_{211\frac{1}{2}} a^\dagger_{211-\frac{1}{2}} ... a^\dagger_{100\frac{1}{2}} a^\dagger_{100-\frac{1}{2}} |0\rangle . \tag{150}$$

A similar interpretation of the spectra of the other alkali atoms can be given. The fact that a quite simple understanding of atoms can be achieved based on the above considerations, indicates that it must indeed be possible to represent the effect of the interaction of the electrons among themselves by an average sp potential. The determination of this average sp potential from the electron-electron interaction will require the explicit consideration of the two-body interaction in *e.g.* a Hartree-Fock procedure. Another simple confirmation of the atomic shell model picture is provided by considering the excited states of the Neon atom which has the $(1s)^2(2s)^2(2p)^6$ configuration occupied in the ground state. All the excited levels can be understood in terms of the promotion of a $2p$-electron to an unoccupied orbital starting with the $3s$, $3p$, $3d$, $4s$, $4p$ and so on. In terms of particle addition and removal operators these states can be obtained from the ground state which is written as $|\Phi_0\rangle$. For example

$$a^\dagger_{3s} a_{2p} |\Phi_0\rangle \tag{151}$$

represents schematically the lowest possible excited states. The presence of more than one energy level at the position of a schematic state like Eq. (151) is due to the splitting that results from the inclusion of the magnetic interactions as well as from the different action of the Coulomb interaction depending on which total angular momentum the resulting configuration has.

C. Independent-particle model for nuclei

The shell structure observed in atoms is also found in nuclei. The origin of shell structure in nuclei is, however, quite different. In addition, it is not as simple as in the atomic case. The shell structure in atoms can be demonstrated by considering the ionization potential. Shell closures at 2, 10, 18, 36, 54, and 86 which signal the position of the noble gas atoms, are then clearly visible. In the case of nuclei a similar quantity, called separation energy, should be considered. For neutrons it is defined by

$$S_n(N,Z) = B(N,Z) - B(N-1,Z) \tag{152}$$

and for protons by

$$S_p(N,Z) = B(N,Z) - B(N,Z-1) \tag{153}$$

where B describes the nuclear binding energy for the nucleus as a function of N, the number of neutrons, and Z, the number of protons. By considering the separation energy a shell closure is visible at $N = 82$. It appears for fixed values of the difference $N - Z$ as a function of N but does display an odd-even staggering that can be interpreted in terms of additional stability of systems with even number of neutrons. A similar feature is observed for protons. One can eliminate this staggering by considering the separation energy only for odd N with Z even, as a function of the number of neutrons or, for protons, for Z odd and N even. Obvious shell closures then occur at N= 50, 82, and 126 and at Z= 50 and 82. Less clear, but deduced from other data, like spectra, are shell closures at N= 2, 8, 20, and 28 as well as Z= 2, 8, 20, and 28.

These "magic" numbers can be related to shell closures as in the atomic case. The sp potential that is responsible for this type of shell structure is generated by the nucleons themselves since there is no center of attraction as in the case of electrons. An empirical sp potential which provides an adequate description of nuclear shell structure is given by

$$U = Vf(r) + V_{ls}(\mathbf{l}\cdot\mathbf{s})r_0^2\frac{1}{r}\frac{d}{dr}f(r) \tag{154}$$

with

$$f(r) = \frac{1}{1+\exp\frac{(r-R)}{a}}. \tag{155}$$

This form is referred to as a Woods-Saxon shape. The constants in this potential are given by the depth

$$V = \left(-51 + 33(\frac{N-Z}{A})\right) \text{ MeV}, \tag{156}$$

the radius parameter

$$R = r_0 A^{1/3}, \tag{157}$$

with

$$r_0 = 1.27\text{fm}, \tag{158}$$

the diffuseness parameter

$$a = 0.67\text{fm}, \tag{159}$$

and the strength of the spin-orbit interaction

$$V_{ls} = -0.44V. \tag{160}$$

A parametrization of this kind is successful in describing the character of the lowest energy states of nuclei with one more or less proton or neutron with respect to the ground-state energy of the doubly magic nucleus ^{208}Pb. In the independent particle model, the ground state of this nucleus is obtained by filling the relevant proton and neutron shells. The lowest energy states for the $A = 209$ system are obtained by putting a proton or neutron in the corresponding lowest available empty shell. The observed and calculated positions of these levels show a good correspondence which is also observed for odd nuclei neighboring other closed-shell nuclei like ^{16}O, ^{40}Ca, ^{48}Ca, and ^{56}Ni. The energy of an additional proton or neutron for an $A = 209$ system is of course in this simple model given by

$$\hat{H}_0 a_\alpha^\dagger \left|{}^{208}\text{Pb}\right\rangle = \left(\epsilon_\alpha + E^{(0)}({}^{208}\text{Pb})\right) a_\alpha^\dagger \left|{}^{208}\text{Pb}\right\rangle \tag{161}$$

where α represents the quantum numbers of an unoccupied proton or neutron state. The experimental information is obtained by subtracting from the ground-state energy of the $A = 209$ system the ground-state energy of ^{208}Pb, which gives the position of the first level either for an extra proton or neutron. The position of the other experimental levels can then be obtained by adding their relative energy to the energy of the ground state of the corresponding $A = 209$ system. One should observe that this procedure allows a comparison with the sp levels obtained from the Woods-Saxon potential although the comparison presupposes that the independent-particle model makes sense. For the states in the $A = 207$ systems one uses

$$\hat{H}_0 a_\alpha \left|{}^{208}\text{Pb}\right\rangle = \left(E^{(0)}({}^{208}\text{Pb}) - \epsilon_\alpha\right) a_\alpha \left|{}^{208}\text{Pb}\right\rangle. \tag{162}$$

The calculated position of the levels corresponds again to ϵ_α and is obtained for the last occupied sp state by subtracting the ground-state energy of the $A = 207$ from $E^{(0)}({}^{208}\text{Pb})$. The experimental information is obtained in a similar way. The higher excited states in the $A = 207$ systems occur therefore lower in energy (are more deeply bound) in this picture.

A Woods-Saxon potential has a finite depth and therefore has a finite number of bound states contrary to a Hydrogen-like potential which has an infinite number of bound states. It also has an exponential fall-off at large r which implies that all bound states are well localized, again contrary to the r^{-1} behavior of a Hydrogen-like potential with the resulting large mean square radii of weakly bound orbitals. The central part of the Woods-Saxon potential can be reasonably approximated by a three-dimensional harmonic oscillator (HO) potential

$$U_{HO}(r) = \frac{1}{2}m\omega^2 r^2 - V_0. \tag{163}$$

The oscillator frequency ω and constant shift V_0 can be adjusted to resemble the Woods-Saxon well in a reasonable way. The HO potential has only discrete eigenstates and care needs to be exercised to interpret the positive energy states which correspond to scattering states for the Woods-Saxon well. The eigenvalues of the HO potential are given by

$$H_{HO}\,|nlm_lm_s\rangle = \big(\hbar\omega(2n+l+3/2) - V_0\big)\,|nlm_lm_s\rangle \tag{164}$$

with

$$\begin{aligned} n &= 0,1,2,...\\ l &= 0,1,2,...\\ -l &< m_l < l. \end{aligned} \tag{165}$$

The total number of oscillator quanta is given by

$$N = 2n + l, \tag{166}$$

which shows that for each oscillator energy only states with the same parity can be degenerate. The HO potential alone leads to magic numbers corresponding to 2, 8, 20, 40, 70, 112, and 168. Although the first three shell closures correspond to experimentally observed shell closures, the other ones show little resemblance to experiment.

A Nobel prize winning suggestion was made by Goeppert-Mayer and Jensen in 1949 [29,30] who introduced, independently, a strong one-body spin-orbit potential (of similar nature as the phenomenological spin-orbit interaction given above). The effect of this potential is mostly felt at the surface of the nucleus since this is where the derivative of $f(r)$ peaks. The presence of the $\mathbf{l}\cdot\mathbf{s}$ operator requires a change of sp basis to states with good total angular momentum

$$|n(ls)jm_j\rangle = \sum_{m_lm_s} |nlm_lsm_s\rangle\,\langle\ l\ m_l\ s\ m_s\ |\ l\ s\ j\ m_j\ \rangle. \tag{167}$$

The transformation bracket is usually referred to as a Clebsch-Gordan coefficient. Using the operator identity

$$\mathbf{l}\cdot\mathbf{s} = \frac{1}{2}(\mathbf{j}^2 - \mathbf{l}^2 - \mathbf{s}^2) \tag{168}$$

one finds

$$\mathbf{l}\cdot\mathbf{s}\,|n(ls)jm_j\rangle = \frac{1}{2}\hbar^2\big(j(j+1) - l(l+1) - \frac{1}{2}(\frac{1}{2}+1)\big)\,|n(ls)jm_j\rangle\,. \tag{169}$$

Obviously for an s-wave the spin-orbit potential will not contribute. For other l-values one obtains for $j = l + \frac{1}{2}$ the eigenvalue $\frac{1}{2}\hbar^2 l$ whereas for $j = l - \frac{1}{2}$ one has $-\frac{1}{2}\hbar^2(l+1)$. Combining this result with the sign of V_{ls} and the sign of the derivative of $f(r)$ one deduces that the introduction of this spin-orbit interaction has the tendency to substantially lower the energy in a given major shell of the subshell with the largest orbital angular momentum and $j = l + \frac{1}{2}$. If this lowering is sufficiently large this orbital can come to reside among the orbitals of the $N-1$ major shell. Additional splitting of the HO degeneracy with the more realistic Woods-Saxon well, will also favor the higher l orbitals. As a result a shell structure is obtained which reflects the experimentally observed shell closures.

The interaction between nucleons can not be considered to be theoretically well understood. In principle one would like to derive the interaction between nucleons from a QCD perspective. This is not possible at this moment. Instead, one can proceed differently and use the experimental data that describe the two-nucleon system and construct interactions that describe these data accurately. Different interactions will then be able to describe these data equally well. This description will thus involve the use of non-relativistic Quantum Mechanics and the corresponding Schrödinger or Lippmann-Schwinger equation. At 140 MeV excitation energy in the two-nucleon system it becomes possible to create an additional pion. These mesonic degrees of freedom are usually not considered explicitly when one is interested in describing nuclear excitations below this threshold. As a result, one can define the nuclear many-body problem by a hamiltonian

$$\hat{H} = \sum_{\alpha\beta}\langle\alpha|\,T\,|\beta\rangle\,a^\dagger_\alpha a_\beta + \frac{1}{2}\sum_{\alpha\beta\gamma\delta}(\alpha\beta|V|\gamma\delta)\ a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma \tag{170}$$

in which one deals with nonrelativistic nucleons and a two-body interaction V which describes all low-energy two-nucleon data in a nonrelativistic framework. Although the long-range part of this interaction can be very fruitfully interpreted in terms of the exchange of a virtual pion (not enough energy to make it real), one can call this an experimental interaction. In commonly used language, such interactions are also characterized as realistic. The nuclear many-body problem is therefore defined but restrictions are obvious: excitations at higher energy than 140 MeV cannot be described realistically. Nevertheless, it will be possible to understand much of the many-particle aspects of the nucleus by considering the above hamiltonian. This does not need to be too surprising since the coupling to the physical states above 140 MeV is, albeit indirectly, experimentally constrained and the medium modifications of the interaction and the properties of nucleons are most sensitive to an energy scale which is related to the Fermi energy.

D. Empirical Mass Formula and Nuclear Matter

Some qualitative aspects of nuclei that will have to be explained on the basis of realistic interactions, are revealed by the systematics of nuclear binding as a function of N and Z and the observation that the density in the interior of nuclei is constant. First the systematics of nuclear binding will be discussed. The total mass of a nucleus with N neutrons and Z protons or its energy can be related to the mass of N neutrons and Z protons by defining the binding energy

$$\begin{aligned} M(N,Z) &= E(N,Z)/c^2 \\ &= N\ m_n\ +\ Z\ m_p\ -\ B(N,Z)/c^2. \end{aligned} \tag{171}$$

A smooth curve through the experimental data for the binding energy is obtained by means of the following expression which is referred to as the semi-empirical mass formula.

$$B = b_{vol}A - b_{surf}A^{2/3} - \frac{1}{2}b_{sym}\frac{(N-Z)^2}{A} - \frac{3}{5}\frac{Z^2e^2}{R_c}. \tag{172}$$

A relevant set of values of the parameters is given by: $b_{vol} = 15.56$ MeV, $b_{surf} = 17.23$ MeV, $b_{sym} = 46.57$ MeV, and $R_c = 1.24A^{1/3}$ fm. Most nuclei have a binding energy of about 8 MeV per particle which is rather small compared to the rest mass of the nucleon which is about 939 MeV. This would suggest that relativity might not be too important although this is a controversial issue. The term proportional to the number of particles in the mass formula is called the volume term. The second term represents the loss of attraction due to the presence of the surface. These first two terms suggest a saturation of the nuclear interaction, implying that nucleons on the average only experience attraction from interacting with other nucleons only at rather short-range. The third term incorporates the tendency of the nuclear force to favor nuclei with $N = Z$ and is called the symmetry energy. The last term represents the energy of a uniformly charged sphere of radius R_c. In the case one could switch off the Coulomb interaction between the protons only the volume term would survive for $N = Z$ in the limit of infinite volume and constant density. This limit is extremely relevant since, as already mentioned, the central density in the interior of nuclei is found to be constant. More precisely: From sophisticated elastic electron scattering experiments one can deduce the charge density at the center of a nucleus; multiplying this number with A/Z one obtains the aforementioned constant central density of 0.16 nucleons per fm^3. The hypothetical system that is obtained in this infinite volume but constant density limit is referred to as nuclear matter. The explanation of the saturation density of nuclear matter and the corresponding binding energy per particle of about 16 MeV, starting from a realistic two-body interaction, is a problem in many-particle physics which continues to be studied to this day. The goal of this nuclear matter problem is to explain why a minimum in the energy per particle of -16 MeV should occur at a density of 0.16 fm^3 and in addition to reproduce these numbers applying many-particle methods. A typical failure up to now has been that when the correct energy at saturation was obtained, this saturation density is about a factor of two too high, or when the correct saturation density was obtained, the energy is only about half of what is required. Relativity or three-body forces have been invoked to cure this unsatisfactory situation.

E. Isospin

Until now it has been mentioned that neutrons and protons display the same magic numbers and therefore the same shell structure. This cannot be an accident. In effect, the mass difference of the neutron and the proton is only about one part in a thousandth of the average of the proton and neutron mass. According to standard Quantum Mechanics, this degeneracy must reflect a symmetry of the hamiltonian describing the strong interaction. In other words, there is an observable which commutes with the hamiltonian H_S and simultaneous eigenstates of this observable and H_S

can be found. Assuming then that the strong force is independent of particle type, one can further neglect the weak and electromagnetic interaction which do distinguish between the proton and the neutron, and eliminate the small mass difference.

In the following discussion [31] an explicit distinction is made between particle addition and removal operators for protons and neutrons. For example the operator $p_\alpha^\dagger$ adds a proton with quantum numbers α to any state in Fock space and the operator $n_\alpha^\dagger$ does the same for a neutron. These operators obey the same anticommutation relations that were studied before

$$\begin{aligned}\{p_\alpha^\dagger, p_\beta\} &= \delta_{\alpha,\beta} \\ \{n_\alpha^\dagger, n_\beta\} &= \delta_{\alpha,\beta}\end{aligned} \tag{173}$$

with all other anticommutators zero including those involving a proton and neutron operator. A Z proton state in this notation is then given by

$$|\alpha_1\alpha_2...\alpha_Z\rangle = p_{\alpha_1}^\dagger p_{\alpha_2}^\dagger ... p_{\alpha_Z}^\dagger |0\rangle, \tag{174}$$

and a state with N neutrons by

$$|\beta_1\beta_2...\beta_N\rangle = n_{\beta_1}^\dagger n_{\beta_2}^\dagger ... n_{\beta_N}^\dagger |0\rangle. \tag{175}$$

A state with Z protons and N neutrons is then given by

$$|\alpha_1\alpha_2...\alpha_Z;\beta_1\beta_2...\beta_N\rangle = p_{\alpha_1}^\dagger p_{\alpha_2}^\dagger ... p_{\alpha_Z}^\dagger n_{\beta_1}^\dagger n_{\beta_2}^\dagger ... n_{\beta_N}^\dagger |0\rangle \tag{176}$$

Observing that at the sp level the interchange of a proton for a neutron with otherwise identical quantum numbers does not change the energy, one can postulate that this should be valid also for any collection of protons and neutrons. The corresponding operators are easily written down in second quantization. The operator which changes neutrons into protons, while leaving all other quantum numbers unchanged, is given by

$$\hat{T}^+ = \sum_\alpha p_\alpha^\dagger n_\alpha \tag{177}$$

and

$$\hat{T}^- = \sum_\alpha n_\alpha^\dagger p_\alpha \tag{178}$$

does the opposite. The assumption is now, based on the degeneracy of the neutron and proton energy, that

$$[\hat{H}_S, \hat{T}^\pm] = 0. \tag{179}$$

Consider now the following commutator of $\hat{T}^+$ and $\hat{T}^-$ denoted by $\hat{T}_3$

$$\begin{aligned}\hat{T}_3 &= \frac{1}{2}[\hat{T}^+, \hat{T}^-] \\ &= \frac{1}{2}[\sum_\alpha p_\alpha^\dagger n_\alpha, \sum_\beta n_\beta^\dagger p_\beta] \\ &= \frac{1}{2}\sum_{\alpha\beta}(p_\alpha^\dagger n_\alpha n_\beta^\dagger p_\beta - n_\beta^\dagger p_\beta p_\alpha^\dagger n_\alpha) \\ &= \frac{1}{2}\sum_{\alpha\beta}(p_\alpha^\dagger n_\alpha n_\beta^\dagger p_\beta - n_\beta^\dagger p_\beta p_\alpha^\dagger n_\alpha + p_\alpha^\dagger n_\beta^\dagger n_\alpha p_\beta - n_\beta^\dagger p_\alpha^\dagger p_\beta n_\alpha) \\ &= \frac{1}{2}\sum_{\alpha\beta}(p_\alpha^\dagger p_\beta \delta_{\alpha,\beta} - n_\beta^\dagger n_\alpha \delta_{\alpha,\beta}) \\ &= \frac{1}{2}\sum_\alpha (p_\alpha^\dagger p_\alpha - n_\alpha^\dagger n_\alpha).\end{aligned} \tag{180}$$

This operator merely counts the number of protons and subtracts the number of neutrons (multiplied by 1/2) and physically one expects

$$[\hat{H}_S, \hat{T}_3] = 0. \tag{181}$$

One can also show that

$$[\hat{T}_3, \hat{T}^{\pm}] = \pm\hat{T}^{\pm}. \tag{182}$$

This implies that these operators satisfy the same algebra as the angular momentum operators. Indeed, defining

$$T_1 = \frac{1}{2}(T^+ + T^-) \tag{183}$$

and

$$T_2 = \frac{1}{2}(T^+ - T^-), \tag{184}$$

there is a one-to-one correspondence between the triplet (T_1, T_2, T_3) and the triplet of angular momentum operators (J_x, J_y, J_z). Since the spectrum of the angular momentum operators $\mathbf{J}^2$ and J_z is solely determined by the commutation relations between J_x, J_y, and J_z, one can simply relabel all these results in terms of new quantum numbers related to the operators $\mathbf{T}^2$ and T_3 which are referred to as total isospin (squared) and its 3-projection. The isospin invariance of the strong interaction means that the strong hamiltonian is invariant under isospin rotations which are generated by the operators T_1, T_2, and T_3 in complete analogy with the angular momentum case. Rotations in iso-space can therefore be written as

$$R(\hat{\mathbf{n}}) = e^{-i\hat{\mathbf{n}}\cdot\mathbf{T}}. \tag{185}$$

Physical states can be labeled with isospin quantum numbers T and M_T, for total isospin and its third component, respectively. Although only states with T_3 as a good quantum number are observed, one can use the full apparatus of angular momentum algebra in making use of the isospin symmetry of the strong interaction. In the example of the proton-neutron doublet one has total isospin $t = 1/2$ and one arbitrarily assigns the proton isospin 3-projection $m_t = 1/2$ and the neutron $m_t = -1/2$. Historically, it was done the other way around in nuclear physics in order to make T_3 positive for nuclei with neutron excess, which represent a vast majority.

Instead of dealing separately with neutrons and protons one can use the isospin formalism. The complete set of nucleon quantum numbers must then include isospin. For example, a proton at $\mathbf{r}$ with spin projection m_s is denoted by

$$|\mathbf{r}m_s\rangle_p = |\mathbf{r}m_s m_t = 1/2\rangle, \tag{186}$$

and similarly for a neutron

$$|\mathbf{r}m_s\rangle_n = |\mathbf{r}m_s m_t = -1/2\rangle. \tag{187}$$

One also has

$$\mathbf{T}^2 |\mathbf{r}m_s m_t\rangle = \frac{1}{2}(\frac{1}{2}+1)|\mathbf{r}m_s m_t\rangle \tag{188}$$

and

$$T_3 |\mathbf{r}m_s m_t\rangle = m_t |\mathbf{r}m_s m_t\rangle. \tag{189}$$

Note that we are actually thinking of particles in an isospin multiplet as identical particles with T_3 as just another label for the state. For this reason the choice was made to let the proton addition and removal operators anticommute with those for neutrons. Examples of the application and usefulness of the isospin concept abound in nuclear physics. Consider the determination of the isospin of a closed-shell. This discussion is similar to the one for angular momentum which will be given here. In order to determine the angular momentum of a closed shell (for example of protons) consider the third component of the total angular momentum operator in second quantization

$$\begin{aligned}\hat{J}_z &= \sum_{n_\alpha l_\alpha j_\alpha m_\alpha} \sum_{n_\beta l_\beta j_\beta m_\beta} \langle n_\alpha l_\alpha j_\alpha m_\alpha | j_z | n_\beta l_\beta j_\beta m_\beta\rangle a^\dagger_{n_\alpha l_\alpha j_\alpha m_\alpha} a_{n_\beta l_\beta j_\beta m_\beta} \\ &= \sum_{n_\alpha l_\alpha j_\alpha m_\alpha} \hbar m_\alpha a^\dagger_{n_\alpha l_\alpha j_\alpha m_\alpha} a_{n_\alpha l_\alpha j_\alpha m_\alpha}.\end{aligned} \tag{190}$$

Without loss of generality one can let this operator act on one full shell where all the particles have quantum numbers n, l, j and all components of j_z are occupied

$$\begin{aligned}
\hat{J}_z \left| nlj; m=-j, m=-j+1, ..., m=j \right\rangle &= \sum_{m_\alpha} \hbar m_\alpha a^\dagger_{n_\alpha l_\alpha j_\alpha m_\alpha} a_{n_\alpha l_\alpha j_\alpha m_\alpha} \left| nlj; m=-j, m=-j+1, ..., m=j \right\rangle \\
&= \left\{ \sum_{m_\alpha=-j}^{j} \hbar m_\alpha \right\} \left| nlj; m=-j, m=-j+1, ..., m=j \right\rangle \\
&= 0 \times \left| nlj; m=-j, m=-j+1, ..., m=j \right\rangle . \qquad (191)
\end{aligned}$$

This result shows that the z-component of the total angular momentum vanishes. By applying the raising and lowering operator to the closed shell in a similar way, one obtains a vanishing result which shows that also the total angular momentum of this state is zero. Similar considerations hold for the total isospin. Note that in this case, for a given shell, both proton and neutron states must be completely filled to yield a total isospin of zero.