VIII. PICTURES

A discussion of the different pictures that are use in Quantum Mechanics for the many-particle problem [1] follows the same steps as in one-particle problems [26]. As discussed in the previous chapter it is possible to use the time-dependent formulation to establish (throught Fourier transformation) insight into the energy properties of the system under study. In addition, it is important to learn to deal with time-dependent interactions to descibe the effect of experimental probes which may transfer energy to the many-particle system.

A. Schrödinger Picture

In this picture one employs the notation

$$|\Psi_S(t)\rangle = |\Psi(t)\rangle \tag{452}$$

to describe the normal time-dependence of a state ket. The Schrödinger equation for this many-particle state reads

$$i\hbar\frac{\partial}{\partial t}|\Psi_S(t)\rangle = \hat{H}\,|\Psi_S(t)\rangle \tag{453}$$

with the initial state $|\Psi_S(t_0)\rangle$ given. For a time-independent hamiltonian one can obtain the state at t from the state at t_0 in the following way

$$\begin{aligned} |\Psi_S(t)\rangle &= \exp\left\{-\frac{i}{\hbar}\hat{H}(t-t_0)\right\}|\Psi_S(t_0)\rangle \\ &= \hat{U}_S(t,t_0)\,|\Psi_S(t_0)\rangle\,, \end{aligned} \tag{454}$$

which defines the time-evolution operator.

B. Interaction Picture

One can express the time-independent hamiltonian of the many-body system in terms of

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \tag{455}$$

where the problem associated with $\hat{H}_0$ is assumed to be completely solved for example in terms of the independent particle model discussed in section #3. In the sp case, the propagator was studied also by considering the propagator belonging to H_0, $G^{(0)}$, to be known so that the exact propagator can be expressed in terms of an expansion involving $G^{(0)}$ and the interaction term V. In the present many-particle context, one proceeds similarly by having the time-dependence governed by $\hat{H}_0$ known so that one can concentrate on the changes with respect to this on account of the action of the interaction term involving $\hat{H}_1$. One defines at any time t

$$|\Psi_I(t)\rangle = \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}|\Psi_S(t)\rangle \tag{456}$$

the interaction picture state ket in terms of the Schrödinger picture state ket and the noninteracting hamiltonian $\hat{H}_0$. The equation of motion for this ket is given by

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle &= -\hat{H}_0\,|\Psi_I(t)\rangle + \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\} i\hbar\frac{\partial}{\partial t}|\Psi_S(t)\rangle \\ &= -\hat{H}_0\,|\Psi_I(t)\rangle + \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\left(\hat{H}_0+\hat{H}_1\right)|\Psi_S(t)\rangle \\ &= \hat{H}_1(t)\,|\Psi_I(t)\rangle\,, \end{aligned} \tag{457}$$

where

$$\hat{H}_1(t) = \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\hat{H}_1\exp\left\{-\frac{i}{\hbar}\hat{H}_0 t\right\}, \tag{458}$$

where one should note that in general $\hat{H}_0$ and $\hat{H}_1$ do not commute. If states are related by Eq. (456), then one may obtain a corresponding relation for operators by considering

$$\hat{O}_S \left|\Psi_S(t)\right\rangle = \left|\Psi'_S(t)\right\rangle \tag{459}$$

so that one can write

$$\begin{aligned}
\left|\Psi'_I(t)\right\rangle &= \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\left|\Psi'_S(t)\right\rangle \\
&= \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\hat{O}_S \left|\Psi_S(t)\right\rangle \\
&= \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\hat{O}_S \exp\left\{-\frac{i}{\hbar}\hat{H}_0 t\right\}\exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\left|\Psi_S(t)\right\rangle \\
&= \hat{O}_I(t)\left|\Psi_I(t)\right\rangle ,
\end{aligned} \tag{460}$$

where

$$\hat{O}_I(t) = \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\hat{O}_S \exp\left\{-\frac{i}{\hbar}\hat{H}_0 t\right\} \tag{461}$$

is the operator in the interaction picture that corresponds to $\hat{O}_S$ in the Schrödinger picture which is usually assumed to have no time dependence. This shows that in the interaction picture both state kets and operators have time dependence noting that this dependence is simple for operators. The equation of motion of an operator in the interaction picture may be obtained by considering

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\hat{O}_I(t) &= \left\{i\hbar\frac{\partial}{\partial t}\exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\right\}\hat{O}_S \exp\left\{-\frac{i}{\hbar}\hat{H}_0 t\right\} + \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\hat{O}_S\left\{i\hbar\frac{\partial}{\partial t}\exp\left\{-\frac{i}{\hbar}\hat{H}_0 t\right\}\right\} \\
&= -\hat{H}_0\hat{O}_I(t) + \hat{O}_I(t)\hat{H}_0 \\
&= \left[\hat{O}_I(t), \hat{H}_0\right] .
\end{aligned} \tag{462}$$

An important example involves the addition (removal) operators of particles with sp quantum numbers corresponding to H_0. In this basis

$$\hat{H}_0 = \sum_{\lambda} \epsilon_\lambda a^\dagger_\lambda a_\lambda . \tag{463}$$

Applying Eq. (462), one obtains

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}a_{\lambda_I}(t) &= \left[a_{\lambda_I}(t), \hat{H}_0\right] \\
&= \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\left[a_\lambda, \hat{H}_0\right]\exp\left\{-\frac{i}{\hbar}\hat{H}_0 t\right\} \\
&= \epsilon_\lambda a_{\lambda_I}(t),
\end{aligned} \tag{464}$$

with the solution

$$a_{\lambda_I}(t) = e^{-i\epsilon_\lambda t/\hbar}a_\lambda \tag{465}$$

and correspondingly

$$a^\dagger_{\lambda_I}(t) = e^{i\epsilon_\lambda t/\hbar}a^\dagger_\lambda . \tag{466}$$

In this basis one therefore obtains for the two-body interaction in the interaction picture

$$\hat{V}_I(t) = \frac{1}{2}\sum_{\alpha\beta\gamma\delta}(\alpha\beta|V|\gamma\delta)\, a^\dagger_{\alpha_I}(t)a^\dagger_{\beta_I}(t)a_{\delta_I}(t)a_{\gamma_I}(t), \tag{467}$$

and similarly for the auxiliary potential (or external field)

$$\hat{U}_I(t) = \sum_{\alpha\beta} (\alpha|U|\beta)\, a^\dagger_{\alpha_I}(t) a_{\beta_I}(t). \tag{468}$$

These results imply that these operators have a simple time-dependence which allows for straightforward time-integrations when Fourier transformations are applied. A special role is played by the time-evolution operator in the interaction picture. One defines

$$|\Psi_I(t)\rangle = \hat{U}(t,t_0)\,|\Psi_I(t_0)\rangle\,, \tag{469}$$

where the I subscript is left out for this special operator and

$$\hat{U}(t_0,t_0) = 1. \tag{470}$$

The explicit construction of this operator is accomplished by considering

$$\begin{aligned} |\Psi_I(t)\rangle &= \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}|\Psi_S(t)\rangle \\ &= \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\exp\left\{-\frac{i}{\hbar}\hat{H}(t-t_0)\right\}|\Psi_S(t_0)\rangle \\ &= \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\exp\left\{-\frac{i}{\hbar}\hat{H}(t-t_0)\right\}\exp\left\{-\frac{i}{\hbar}\hat{H}_0 t_0\right\}|\Psi_I(t_0)\rangle\,, \end{aligned} \tag{471}$$

which shows that

$$\hat{U}(t,t_0) = \exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\exp\left\{-\frac{i}{\hbar}\hat{H}(t-t_0)\right\}\exp\left\{-\frac{i}{\hbar}\hat{H}_0 t_0\right\}, \tag{472}$$

where again it must be emphasized that $\hat{H}$ and $\hat{H}_0$ do not commute. Using this result one can show the following property of the time-evolution operator in the interaction picture that will be useful later on

$$\hat{U}^\dagger(t,t_0)\hat{U}(t,t_0) = \hat{U}(t,t_0)\hat{U}^\dagger(t,t_0) = 1 \tag{473}$$

demonstrating unitarity

$$\hat{U}^\dagger(t,t_0) = \hat{U}^{-1}(t,t_0). \tag{474}$$

One also has

$$\hat{U}(t_1,t_2)\hat{U}(t_2,t_3) = \hat{U}(t_1,t_3) \tag{475}$$

and

$$\hat{U}(t,t_0)\hat{U}(t_0,t) = 1, \tag{476}$$

which shows that

$$\hat{U}(t_0,t) = \hat{U}^\dagger(t,t_0). \tag{477}$$

For practical applications it is also important to consider an alternative expression for $\hat{U}$. By combining Eqs. (457) and (469) one obtains

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_1(t)\hat{U}(t,t_0). \tag{478}$$

Using the boundary condition given by Eq. (470) one can formally integrate this operator equation with the result

$$\hat{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\ \hat{H}_1(t')\hat{U}(t',t_0). \tag{479}$$

One can iterate this equation to obtain an expansion in terms of $\hat{H}_1$

$$
\begin{aligned}
\hat{U}(t,t_0) &= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\ \hat{H}_1(t')\left\{1 - \frac{i}{\hbar}\int_{t_0}^{t'} dt''\ \hat{H}_1(t'')\hat{U}(t'',t_0)\right\} \\
&= 1 + \left(\frac{-i}{\hbar}\right)\int_{t_0}^{t} dt'\ \hat{H}_1(t') + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\ \hat{H}_1(t')\hat{H}_1(t'') + ... \\
&= \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 ... \int_{t_0}^{t_{n-1}} dt_n\ \hat{H}_1(t_1)\hat{H}_1(t_2)...\hat{H}_1(t_n).
\end{aligned}
\tag{480}
$$

One can rewrite the second order term in the following way

$$
\begin{aligned}
\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\ \hat{H}_1(t')\hat{H}_1(t'') &= \frac{1}{2}\left\{\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\ \hat{H}_1(t')\hat{H}_1(t'') + \int_{t_0}^{t} dt'' \int_{t''}^{t} dt'\ \hat{H}_1(t')\hat{H}_1(t'')\right\} \\
&= \frac{1}{2}\left\{\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\ \hat{H}_1(t')\hat{H}_1(t'') + \int_{t_0}^{t} dt' \int_{t'}^{t} dt''\ \hat{H}_1(t'')\hat{H}_1(t')\right\} \\
&= \frac{1}{2}\left\{\int_{t_0}^{t} dt' \int_{t_0}^{t} dt''\ \left[\theta(t'-t'')\hat{H}_1(t')\hat{H}_1(t'') + \theta(t''-t')\hat{H}_1(t'')\hat{H}_1(t')\right]\right\} \\
&= \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt''\ \mathcal{T}\left[\hat{H}_1(t')\hat{H}_1(t'')\right],
\end{aligned}
\tag{481}
$$

where the time-ordering operation in the last equality is denoted by $\mathcal{T}$. In this development one uses the option to integrate first over t' and then over t'', one interchanges then in this term t' and t'', and finally uses step functions to extend the integration interval in both cases from t_0 to t. With a little puzzling one can convince oneself that this result can be extended to any order. The resulting expansion of $\hat{U}$ therefore can be written as

$$
\hat{U}(t,t_0) = \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 ... \int_{t_0}^{t} dt_n\ \mathcal{T}\left[\hat{H}_1(t_1)\hat{H}_1(t_2)...\hat{H}_1(t_n)\right],
\tag{482}
$$

where the $\mathcal{T}$-operation is extended to order the operator with the latest time farthest to the left, and so on.

C. Heisenberg Picture

One can make the state kets independent of time while assigning time-dependence, governed by the full hamiltonian, to the operators by employing the Heisenberg picture. One defines

$$
|\Psi_H(t)\rangle = \exp\left\{\frac{i}{\hbar}\hat{H}t\right\}|\Psi_S(t)\rangle,
\tag{483}
$$

using the full hamiltonian. It is immediately clear then that

$$
i\hbar\frac{\partial}{\partial t}|\Psi_H(t)\rangle = -\hat{H}\,|\Psi_H(t)\rangle + \hat{H}\,|\Psi_H(t)\rangle = 0,
\tag{484}
$$

which confirms that state kets do not depend on time ($|\Psi_H(t)\rangle = |\Psi_H\rangle$). For operators one can first consider again

$$
\hat{O}_S\,|\Psi_S(t)\rangle = |\Psi'_S(t)\rangle
\tag{485}
$$

so that one can write

$$
\begin{aligned}
|\Psi'_H\rangle &= \exp\left\{\frac{i}{\hbar}\hat{H}t\right\}|\Psi'_S(t)\rangle \\
&= \exp\left\{\frac{i}{\hbar}\hat{H}t\right\}\hat{O}_S \exp\left\{-\frac{i}{\hbar}\hat{H}t\right\}\exp\left\{\frac{i}{\hbar}\hat{H}t\right\}|\Psi_S(t)\rangle \\
&= \hat{O}_H(t)\,|\Psi_H\rangle,
\end{aligned}
\tag{486}
$$

where

$$\hat{O}_H(t) = \exp\left\{\frac{i}{\hbar}\hat{H}t\right\}\hat{O}_S \exp\left\{-\frac{i}{\hbar}\hat{H}t\right\} \tag{487}$$

is the operator in the Heisenberg picture that corresponds to $\hat{O}_S$. The equation of motion of an operator in the Heisenberg picture may be obtained by considering

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\hat{O}_H(t) &= \left\{i\hbar\frac{\partial}{\partial t}\exp\left\{\frac{i}{\hbar}\hat{H}t\right\}\right\}\hat{O}_S \exp\left\{-\frac{i}{\hbar}\hat{H}t\right\} + \exp\left\{\frac{i}{\hbar}\hat{H}t\right\}\hat{O}_S\left\{i\hbar\frac{\partial}{\partial t}\exp\left\{-\frac{i}{\hbar}\hat{H}t\right\}\right\} \\
&= -\hat{H}\hat{O}_H(t) + \hat{O}_H(t)\hat{H} \\
&= \left[\hat{O}_H(t), \hat{H}\right] \\
&= \exp\left\{\frac{i}{\hbar}\hat{H}t\right\}\left[\hat{O}_S, \hat{H}\right]\exp\left\{-\frac{i}{\hbar}\hat{H}t\right\},
\end{aligned} \tag{488}$$

which shows that if a Schrödinger operator commutes with the hamiltonian the corresponding Heisenberg operator is a constant of motion. The relation between operators in the interaction and Heisenberg picture can be obtained by using the relation of both operators to the corresponding one in the Schrödinger picture

$$\begin{aligned}
\hat{O}_H(t) &= \exp\left\{\frac{i}{\hbar}\hat{H}t\right\}\hat{O}_S \exp\left\{-\frac{i}{\hbar}\hat{H}t\right\} \\
&= \exp\left\{\frac{i}{\hbar}\hat{H}t\right\}\exp\left\{-\frac{i}{\hbar}\hat{H}_0 t\right\}\hat{O}_I(t)\exp\left\{\frac{i}{\hbar}\hat{H}_0 t\right\}\exp\left\{-\frac{i}{\hbar}\hat{H}t\right\} \\
&= \hat{U}(0,t)\hat{O}_I(t)\hat{U}(t,0).
\end{aligned} \tag{489}$$

At $t = 0$ state kets in the different pictures coincide

$$|\Psi_H\rangle = |\Psi_S(t=0)\rangle = |\Psi_I(t=0)\rangle \tag{490}$$

as well as the operators

$$\hat{O}_S = \hat{O}_H(t=0) = \hat{O}_I(t=0). \tag{491}$$

For stationary solutions (energy eigenstates) one has in addition

$$\begin{aligned}
|\Psi_{n_S}(t)\rangle &= e^{-iE_n t/\hbar}|\Psi_n\rangle \\
&= e^{-i\hat{H}t/\hbar}|\Psi_n\rangle,
\end{aligned} \tag{492}$$

which shows that the corresponding states in the Heisenberg picture satisfy the time-independent form of the Schrödinger equation and

$$|\Psi_n\rangle = |\Psi_{n_H}\rangle. \tag{493}$$

For the development in the next section it is also useful to observe that

$$|\Psi_H\rangle = |\Psi_I(0)\rangle = \hat{U}(0,t_0)|\Psi_I(t_0)\rangle, \tag{494}$$

which allows one to construct exact eigenstates from interaction picture state kets at an earlier time t_0. This property will be used in the following by using the formal mechanism of the adiabatic switching on of the interaction.

D. The adiabatic switching one of the full interaction

This formal procedure can be used to obtain expressions for exact eigenstates of the full hamiltonian from those of the noninteracting hamiltonian in terms of an expansion in $\hat{H}_1$. Starting in the Schrödinger picture, one defines the time-dependent hamiltonian

$$\hat{H}_S^\epsilon(t) = \hat{H}_0 + e^{-\epsilon|t|}\hat{H}_1, \tag{495}$$

where ϵ is the usual infinitesimally small positive quantity. This hamiltonian reduces to $\hat{H}_0$ when $t \to \pm\infty$, yielding a solvable problem, while at $t = 0$ it coincides with the full hamiltonian. For a given ϵ, one therefore has a means to turn on the full hamiltonian"slowly." Meaningful results cannot depend on the choice of ϵ and one needs to investigate with some care the limit of $\epsilon \to 0$ which represents the infinitely slow limit. First one can observe that the time-evolution operator (in the interaction picture corresponding to the new "$\hat{H}_1$" can be obtained immediately from Eq. (482)

$$\hat{U}_\epsilon(t,t_0) = \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 ... \int_{t_0}^{t} dt_n \; e^{-\epsilon(|t_1|+|t_2|+...+|t_n|)}\mathcal{T}\left[\hat{H}_1(t_1)\hat{H}_1(t_2)...\hat{H}_1(t_n)\right], \tag{496}$$

since the explicit time-dependence introduced in Eq. (495) commutes will all relevant terms. To obtain Eq. (496) one therefore proceeds as in the previous section or directly replace $\hat{H}_1(t_i)$ by $\exp(-\epsilon|t_i|)\hat{H}_1(t_i)$. One may then consider the result corrsponding to Eq. (494)

$$|\Psi_H^\epsilon\rangle = |\Psi_I^\epsilon(0)\rangle = \hat{U}_\epsilon(0,t_0)\,|\Psi_I^\epsilon(t_0)\rangle \tag{497}$$

and consider the limit $t_0 \to -\infty$. In this limit the hamiltonian becomes the noninteracting one and therefore the Schrödinger picture state ket in this limit that evolves from an eigenket $|\Phi_0\rangle$ of $\hat{H}_0$

$$\hat{H}_0\,|\Phi_0\rangle = E_0\,|\Phi_0\rangle\,, \tag{498}$$

can be written as

$$|\Psi_{0_S}(t_0 \to -\infty)\rangle = e^{-iE_0t_0/\hbar}\,|\Phi_0\rangle\,, \tag{499}$$

which corresponds to a stationary state at this time and is independent of ϵ. The corresponding interaction picture ket is given by

$$|\Psi_{0_I}(t_0 \to -\infty)\rangle = e^{i\hat{H}_0t_0/\hbar}\,|\Psi_{0_S}(t_0 \to -\infty)\rangle = |\Phi_0\rangle\,. \tag{500}$$

With this result one can write Eq. (497) as

$$\left|\Psi_{0_H}^\epsilon\right\rangle = \left|\Psi_{0_I}^\epsilon(0)\right\rangle = \hat{U}_\epsilon(0,t_0 \to -\infty)\,|\Phi_0\rangle \tag{501}$$

Usually one starts from the noninteracting ground state and this choice will be made here also although it is not required. The explicit ϵ-dependence in Eq. (501) signals that one now has to study the limit $\epsilon \to 0$ and clarify when meaningful results are obtained. In particular, one is interested in adiabatically evolving the interacting ground state from the noninteracting one.

E. Gell-Mann and Low Theorem

The Gell-Mann and Low theorem [57] asserts that if

$$\lim_{\epsilon\to 0}\frac{\hat{U}_\epsilon(0,-\infty)\,|\Phi_0\rangle}{\langle\Phi_0|\,\hat{U}_\epsilon(0,-\infty)\,|\Phi_0\rangle} = \lim_{\epsilon\to 0}\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle} \equiv \frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} \tag{502}$$

exists to all orders in perturbation theory, then this state is an eigen ket of the full hamiltonian,

$$\hat{H}\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = E\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}, \tag{503}$$

where

$$E - E_0 = \frac{\langle\Phi_0|\,\hat{H}_1\,|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}, \tag{504}$$

which can be obtained by multiplying Eq. (503) from the left by $\langle\Phi_0|$. This last equation is useful to construct a time-dependent analysis of this energy difference also called correlation energy [58,59], which involves socalled Goldstone diagrams. It is important to realize that the numerator and denominator of Eq. (502) do not exist separately when $\epsilon \to 0$. The first two terms in the numerator for example are given by

$$\begin{aligned} \hat{U}_\epsilon(0,-\infty)\,|\Phi_0\rangle &= 1 + \left(\frac{-i}{\hbar}\right)\int_{-\infty}^{0} dt_1\; e^{\epsilon t_1}\hat{H}_1(t_1)\,|\Phi_0\rangle + ... \\ &= 1 + \left(\frac{-i}{\hbar}\right)\int_{-\infty}^{0} dt_1\; e^{\epsilon t_1}\frac{1}{2}\sum_{\alpha\beta\gamma\delta}(\alpha\beta|V|\gamma\delta)\,e^{i(\epsilon_\alpha+\epsilon_\beta-\epsilon_\delta-\epsilon_\gamma)t_1/\hbar}a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma\,|\Phi_0\rangle + ... \end{aligned} \tag{505}$$

Performing the integration for those cases where $\epsilon_\alpha + \epsilon_\beta = \epsilon_\gamma + \epsilon_\delta$, demonstrates that this term contains a $1/\epsilon$ dependence which diverges for $\epsilon \to 0$. Therefore one must expect a cancellation to take place between numerator and denominator such that infinities like these have already been eliminated before the limit is taken.

The development of the proof requires considering

$$\left(\hat{H}_0 - E_0\right)|\Psi_0^\epsilon\rangle = \left(\hat{H}_0 - E_0\right)\hat{U}_\epsilon(0,-\infty)\,|\Phi_0\rangle = \left[\hat{H}_0, \hat{U}_\epsilon(0,-\infty)\right]|\Phi_0\rangle\,, \tag{506}$$

where the picture designation for the state ket at $t = 0$ has been eliminated since all pictures coincide at this time. In evaluating the commutator in Eq. (506) one uses the explicit form of $\hat{U}_\epsilon$ given in Eq. (496). In evaluating the commutator one has to consider in nth order and for a particular time-ordering

$$\begin{aligned} \left[\hat{H}_0, \hat{H}_1(t_{p_1})...\hat{H}_1(t_{p_j})...\hat{H}_1(t_{p_n})\right] &= \left[\hat{H}_0, \hat{H}_1(t_{p_1})\right]\hat{H}_1(t_{p_2})...\hat{H}_1(t_{p_n}) \\ &+ ... + \hat{H}_1(t_{p_1})...\left[\hat{H}_0, \hat{H}_1(t_{p_j})\right]...\hat{H}_1(t_{p_n}) + ... \\ &+ \hat{H}_1(t_{p_1})...\hat{H}_1(t_{p_j})...\left[\hat{H}_0, \hat{H}_1(t_{p_n})\right]. \end{aligned} \tag{507}$$

One can then use

$$i\hbar\frac{\partial}{\partial t}\hat{H}_1(t) = \left[\hat{H}_1(t), \hat{H}_0\right] \tag{508}$$

for each of these commutators so that

$$\left[\hat{H}_0, \hat{H}_1(t_{p_1})...\hat{H}_1(t_{p_j})...\hat{H}_1(t_{p_n})\right] = \frac{\hbar}{i}\left(\frac{\partial}{\partial t_{p_1}} + ... + \frac{\partial}{\partial t_{p_n}}\right)\hat{H}_1(t_{p_1})...\hat{H}_1(t_{p_j})...\hat{H}_1(t_{p_n}) \tag{509}$$

This result is true for any time-ordering. For the present time-ordering one also has

$$\begin{aligned} \theta(t_{p_1} - t_{p_2})...\theta(t_{p_{n-1}} - t_{p_n}) \quad & \frac{\hbar}{i}\left\{\sum_{i=1}^{n}\frac{\partial}{\partial t_{p_i}}\right\}\left[\hat{H}_1(t_{p_1})...\hat{H}_1(t_{p_n})\right] \\ &= \frac{\hbar}{i}\left\{\sum_{i=1}^{n}\frac{\partial}{\partial t_{p_i}}\right\}\left[\theta(t_{p_1} - t_{p_2})...\theta(t_{p_{n-1}} - t_{p_n})\hat{H}_1(t_{p_1})...\hat{H}_1(t_{p_n})\right], \end{aligned} \tag{510}$$

on account of

$$\left\{\sum_{i=1}^{n}\frac{\partial}{\partial t_{p_i}}\right\}\left[\theta(t_{p_1} - t_{p_2})\theta(t_{p_2} - t_{p_3})...\theta(t_{p_{n-1}} - t_{p_n})\right] = 0, \tag{511}$$

which may be verified by using Eq. (389). Using this result, which holds for any time-ordering, in Eq. (506), allows the sum of the time derivatives to be taken outside of the time-ordered product

$$\mathcal{T}\left[\left\{\sum_{i=1}^{n}\frac{\partial}{\partial t_i}\right\}\hat{H}_1(t_1)...\hat{H}_1(t_n)\right] = \left\{\sum_{i=1}^{n}\frac{\partial}{\partial t_i}\right\}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right]. \tag{512}$$

Collecting the results of Eqs. (512) and (506) one obtains

$$\begin{aligned} \left(\hat{H}_0 - E_0\right)|\Psi_0^\epsilon\rangle &= -\sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^{n-1}\frac{1}{n!}\int_{-\infty}^{0}dt_1...\int_{-\infty}^{0}dt_n\; e^{-\epsilon(|t_1|+...+|t_n|)}\left\{\sum_{i=1}^{n}\frac{\partial}{\partial t_i}\right\}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right]|\Phi_0\rangle \\ &= -\sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^{n-1}\frac{1}{(n-1)!}\int_{-\infty}^{0}dt_1...\int_{-\infty}^{0}dt_n\; e^{-\epsilon(|t_1|+...+|t_n|)}\frac{\partial}{\partial t_1}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right]|\Phi_0\rangle\,, \end{aligned} \tag{513}$$

where in the last step one observes that each time derivative gives an identical contribution so one can relabel, multiply by n, and keep only the $\partial/\partial t_1$ term. The next step involves an integration by parts in the variable t_1 as follows

$$\begin{aligned}\int_{-\infty}^{0} dt_1\; e^{-\epsilon(|t_1|)}\frac{\partial}{\partial t_1}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right] &= e^{-\epsilon(|t_1|)}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right]\Big|_{-\infty}^{0} - \epsilon\int_{-\infty}^{0} dt_1\; e^{-\epsilon(|t_1|)}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right] \\ &= \mathcal{T}\left[\hat{H}_1(0)...\hat{H}_1(t_n)\right] - \epsilon\int_{-\infty}^{0} dt_1\; e^{-\epsilon(|t_1|)}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right]. \end{aligned} \tag{514}$$

In the first term on the right-hand side one can now move $\hat{H}_1$ in front of the time-ordering operation and, subsequently, insert Eq. (514) into (513) with the result

$$\begin{aligned}\left(\hat{H}_0 - E_0\right)|\Psi_0^\epsilon\rangle &= -\hat{H}_1\left\{\sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^{n-1}\frac{1}{(n-1)!}\int_{-\infty}^{0} dt_2...\int_{-\infty}^{0} dt_n\; e^{-\epsilon(|t_2|+...+|t_n|)}\mathcal{T}\left[\hat{H}_1(t_2)...\hat{H}_1(t_n)\right]|\Phi_0\rangle\right\} \\ &+ \epsilon\sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^{n-1}\frac{1}{(n-1)!}\int_{-\infty}^{0} dt_1...\int_{-\infty}^{0} dt_n\; e^{-\epsilon(|t_1|+...+|t_n|)}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right]|\Phi_0\rangle. \end{aligned} \tag{515}$$

In the first term one recognizes between the curly brackets the ket $|\Psi)0^\epsilon\rangle$. To realize the meaning of the second term it is useful to rewrite

$$\hat{H}_1(t) = \lambda\hat{\mathcal{H}}_1(t), \tag{516}$$

where λ acts as a coupling constant that may be varied. The n coupling constants in the second term can then be manipulated as follows

$$\left(\frac{-i}{\hbar}\right)^{n-1}\frac{1}{(n-1)!}\lambda^n = i\hbar\lambda\frac{\partial}{\partial\lambda}\left(\frac{-i}{\hbar}\right)^{n}\frac{1}{n!}\lambda^n, \tag{517}$$

so the second term in Eq. (515) can be written as

$$\begin{aligned}&\epsilon\sum_{n=1}^{\infty}\left(\frac{-i}{\hbar}\right)^{n-1}\frac{1}{(n-1)!}\int_{-\infty}^{0} dt_1...\int_{-\infty}^{0} dt_n\; e^{-\epsilon(|t_1|+...+|t_n|)}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right]|\Phi_0\rangle \\ &= i\hbar\lambda\epsilon\frac{\partial}{\partial\lambda}\left\{\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^{n}\frac{1}{n!}\int_{-\infty}^{0} dt_1...\int_{-\infty}^{0} dt_n\; e^{-\epsilon(|t_1|+...+|t_n|)}\mathcal{T}\left[\hat{H}_1(t_1)...\hat{H}_1(t_n)\right]|\Phi_0\rangle\right\} \\ &= i\hbar\lambda\epsilon\frac{\partial}{\partial\lambda}|\Psi_0^\epsilon\rangle. \end{aligned} \tag{518}$$

With this result one can therefore write

$$\left(\hat{H}_0 - E_0\right)|\Psi_0^\epsilon\rangle = -\hat{H}_1|\Psi_0^\epsilon\rangle + i\hbar\lambda\epsilon\frac{\partial}{\partial\lambda}|\Psi_0^\epsilon\rangle. \tag{519}$$

Rearranging this result into

$$\left(\hat{H} - E_0\right)|\Psi_0^\epsilon\rangle = i\hbar\lambda\epsilon\frac{\partial}{\partial\lambda}|\Psi_0^\epsilon\rangle \tag{520}$$

emphasizes that the limit $\epsilon \to 0$ at this point is nonsensical. It is now time to consider both numerator and denominator of Eq. (502) in the following form

$$\begin{aligned}i\hbar\epsilon\lambda\frac{\partial}{\partial\lambda}\left\{\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\right\} &= i\hbar\epsilon\lambda\frac{1}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\frac{\partial}{\partial\lambda}|\Psi_0^\epsilon\rangle + i\hbar\epsilon\lambda|\Psi_0^\epsilon\rangle\left\{\frac{-1}{\langle\Phi_0|\Psi_0^\epsilon\rangle^2}\right\}\frac{\partial}{\partial\lambda}\langle\Phi_0|\Psi_0^\epsilon\rangle \\ &= i\hbar\epsilon\lambda\frac{1}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\frac{\partial}{\partial\lambda}|\Psi_0^\epsilon\rangle - i\hbar\epsilon\lambda\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\frac{\partial}{\partial\lambda}\ln\langle\Phi_0|\Psi_0^\epsilon\rangle \end{aligned} \tag{521}$$

This result can be rewritten as

$$i\hbar\epsilon\lambda\frac{1}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\frac{\partial}{\partial\lambda}|\Psi_0^\epsilon\rangle = i\hbar\epsilon\lambda\frac{\partial}{\partial\lambda}\left\{\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\right\} + i\hbar\epsilon\lambda\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\frac{\partial}{\partial\lambda}\ln\langle\Phi_0|\Psi_0^\epsilon\rangle. \tag{522}$$

Dividing Eq. (520) by $\langle\Phi_0|\Psi_0^\epsilon\rangle$ and using Eq. (522) one obtains

$$\left(\hat{H} - E_0\right)\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle} = i\hbar\epsilon\lambda\frac{\partial}{\partial\lambda}\left\{\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\right\} + i\hbar\epsilon\lambda\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\frac{\partial}{\partial\lambda}\ln\langle\Phi_0|\Psi_0^\epsilon\rangle. \tag{523}$$

The next step involves mulitplying Eq. (520) from the left by the bra $\langle\Phi_0|/\langle\Phi_0|\Psi_0^\epsilon\rangle$ which yields

$$\frac{\langle\Phi_0|\,\hat{H}_1\,|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle} = \frac{i\hbar\epsilon\lambda}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\frac{\partial}{\partial\lambda}\langle\Phi_0|\Psi_0^\epsilon\rangle = i\hbar\epsilon\lambda\frac{\partial}{\partial\lambda}\ln\langle\Phi_0|\Psi_0^\epsilon\rangle. \tag{524}$$

The term with the logarithm in Eq. (524) also occurs in Eq. (523). Upon substitution of (524) in (523) one then has

$$\left(\hat{H} - E_0\right)\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle} = i\hbar\epsilon\lambda\frac{\partial}{\partial\lambda}\left\{\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\right\} + \frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\frac{\langle\Phi_0|\,\hat{H}_1\,|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}, \tag{525}$$

which can be rearranged to read

$$\left\{\hat{H} - \frac{\langle\Phi_0|\,\hat{H}\,|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\right\}\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle} = i\hbar\epsilon\lambda\frac{\partial}{\partial\lambda}\left\{\frac{|\Psi_0^\epsilon\rangle}{\langle\Phi_0|\Psi_0^\epsilon\rangle}\right\}. \tag{526}$$

Finally, one can now take the limit $\epsilon \to 0$, while using the original assertion that the term in brackets on the right side of Eq. (526) is finite in every order of perturbation theory, to finalize the proof of the Gell-Mann Low theorem

$$\left\{\hat{H} - \frac{\langle\Phi_0|\,\hat{H}\,|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}\right\}\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = 0, \tag{527}$$

or

$$\hat{H}\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle} = E\frac{|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}, \tag{528}$$

with

$$E = \frac{\langle\Phi_0|\,\hat{H}\,|\Psi_0\rangle}{\langle\Phi_0|\Psi_0\rangle}. \tag{529}$$

One can repeat this proof by considering the evolution from $|\Phi_0\rangle$ at $t = \infty$ to $t = 0$. If the resulting state is degenerate with the one obtained in Eq. (528), it must be the same state. Note, however, that the use of the ground state of $\hat{H}_0$ as a starting point does not guarantee that the resulting $|\Psi_0\rangle$ has to be the interacting ground state. The choice of the auxiliary potential provides a tool to construct an appropriate starting point for this evolution process in the case of symmetry breaking. An example is provided by a correlated state that has a net magnetic moment, while the hamiltonian itself does not break rotational symmetry. Nevertheless, the correlated ground state, so the state with the lowest energy, may have the net number of spins in a particular direction be larger than the number pointing in the opposite direction. The exchange term in the two-body matrix element of the Coulomb interaction is responsible for this possibility [18]. The auxiliary potential may then be chosen to favor one spin projection over an other and the corresponding noninteracting ground state will provide a better starting point for considerations based on perturbation theory.