IV. FERMI GAS AND TWO-BODY INTERACTIONS

A. The Fermi gas at zero temperature

For an important class of systems a good starting point is the Fermi gas. It is instructive to consider this system first at zero temperature. Such an idealized system contains fermions with no mutual interaction with each particle only having kinetic energy

$$H = T = \frac{\mathbf{p}^2}{2m}. \tag{192}$$

Eigenstates of Eq. (192) are momentum eigentates. If we assume spin 1/2 fermions, one has

$$\frac{\mathbf{p}^2}{2m} \left| \mathbf{p}' \; m_s \right\rangle = \frac{\mathbf{p}'^2}{2m} \left| \mathbf{p}' \; m_s \right\rangle . \tag{193}$$

In general one can label the discrete spin and/or isospin quantum numbers with an index μ. This includes the possibility that only one (spin/isospin) species of fermions is contained in the system. If we consider the momentum states in a box with sides L and volume $V = L^3$ one can write the sp wave function as

$$\langle \mathbf{x}' \; m_{s'} | \mathbf{p} \; m_s \rangle = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}'} \delta_{m_s, m_{s'}}. \tag{194}$$

Normalization can be chosen as follows

$$\langle \mathbf{p}' \; m_{s'} | \mathbf{p} \; m_s \rangle = \delta_{\mathbf{p}',\mathbf{p}} \delta_{m_{s'}, m_s}. \tag{195}$$

Suppressing spin and using the wave function (Eq. (194)) this translates into

$$\langle \mathbf{p}' | \mathbf{p} \rangle = \int_{box} d\mathbf{x} \; \langle \mathbf{p}' | \mathbf{x} \rangle \langle \mathbf{x} | \mathbf{p} \rangle = \frac{1}{V} \int d\mathbf{x} \; e^{\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} = \delta_{\mathbf{p}',\mathbf{p}}. \tag{196}$$

Usually one is interested in the situation where the size of the box goes to infinity ($L \to \infty$). Before doing this it is convenient to consider periodic boundary conditions as suggested by translational invariance. In the x-direction for example one requires (introducing $p_x = \hbar k_x$)

$$e^{ik_x x} = e^{ik_x(x+L)} = e^{ik_x x} e^{ik_x L} \tag{197}$$

so

$$\cos(k_x + L) + i \sin(k_x + L) = 1, \tag{198}$$

which is fulfilled when

$$k_x = n_x \frac{2\pi}{L} \qquad\qquad n_x = 0, \pm 1, \pm 2, ... \tag{199}$$

and similarly for k_y and k_z. This means that each triple $\{k_x, k_y, k_z\}$ corresponds to a triple of integers $\{n_x, n_y, n_z\}$. Since the Pauli principle allows only a fixed number of fermions in each sp momentum eigenstate (depending on the spin and/or isospin degeneracy), the ground state is obtained by filling the momentum states up to a maximum value, the Fermi momentum $p_F = \hbar k_F$. The maximum wavenumber can be determined by calculating the expectation value of the number operator (see Eq. (103)) in the ground state

$$|\Phi_0\rangle = \prod_{\mathbf{p}\mu, p<\hbar k_F} a^\dagger_{\mathbf{p}\mu} |0\rangle . \tag{200}$$

In order to obtain this result, it is now useful to consider the limit when both $N \to \infty$ and $V \to \infty$ in such a way that their ratio (the density) remains constant. This limit is sometimes referred to as the "thermodynamic limit." Sums over states can now be replaced by integrations over continuous quantum numbers in the following way

$$\sum_{\mathbf{k}\mu} f(\mathbf{k},\mu) = \sum_{n_x n_y n_z} \sum_{\mu} f(\frac{2\pi\mathbf{n}}{L},\mu) \to_{thermodynamic\ limit} \int d\mathbf{n} \sum_{\mu} f(\frac{2\pi\mathbf{n}}{L},\mu) = \frac{V}{(2\pi)^3} \sum_{\mu} \int d\mathbf{k} f(\mathbf{k},\mu) \tag{201}$$

for any function f. The transition from discrete triples $\{n_x, n_y, n_z\}$ to continuous variables can be made in the case of large L since any physical quantity described by f will change slowly when one of the discrete variables changes by one unit. To obtain the Fermi wavenumber consider

$$N = \langle\Phi_0|\,\hat{N}\,|\Phi_0\rangle = \sum_{\mathbf{k}\mu} \langle\Phi_0|\, a^\dagger_{\mathbf{k}\mu} a_{\mathbf{k}\mu}\,|\Phi_0\rangle = \sum_{\mathbf{k}\mu} \theta(k_F - k)$$

$$= \frac{V}{(2\pi)^3}\sum_\mu \int d^3k\,\theta(k_F - k) = \frac{\nu V}{6\pi^2} k_F^3, \tag{202}$$

where ν represents the spin/isospin degeneracy and θ denotes the step function. The relation between the Fermi wavenumber and the density therefore becomes

$$k_F = \left\{\frac{6\pi^2 N}{\nu V}\right\}^{1/3} = \left\{\frac{9\pi}{2\nu}\right\}^{1/3} \frac{1}{r_0}, \tag{203}$$

where in the last equality r_0 has been introduced which is obtained from the volume per particle

$$\frac{V}{N} = \frac{1}{\rho} = \frac{4}{3}\pi r_0^3. \tag{204}$$

Clearly r_0 also serves as a measure of the interparticle spacing. Conversely one can write the density as

$$\rho = \frac{N}{V} = \nu \frac{k_F^3}{6\pi^2}. \tag{205}$$

Eqs. (203) and (205) show that for a fixed density one has a smaller Fermi wavenumber when the degeneracy factor ν is larger.

The energy of the ground state of the Fermi gas is obtained by employing the kinetic energy operator

$$\hat{T} = \sum_{\mathbf{p}\mu}\sum_{\mathbf{p}'\mu'} \langle\mathbf{p}\mu|\,\frac{\mathbf{p}^2}{2m}\,|\mathbf{p}'\mu'\rangle\, a^\dagger_{\mathbf{p}\mu} a_{\mathbf{p}'\mu'} = \sum_{\mathbf{p}'\mu} \frac{\mathbf{p}'^2}{2m} a^\dagger_{\mathbf{p}'\mu} a_{\mathbf{p}'\mu}. \tag{206}$$

One thus has to calculate

$$\hat{T}\,|\Phi_0\rangle = \left(\sum_{\mathbf{p}'\mu} \frac{\mathbf{p}'^2}{2m} a^\dagger_{\mathbf{p}'\mu} a_{\mathbf{p}'\mu}\right) \prod_{\mathbf{p}\mu,\, p<\hbar k_F} a^\dagger_{\mathbf{p}\mu}\,|0\rangle\,. \tag{207}$$

Using Eq. (101) one obtains a kinetic energy contribution from each sp state that is occupied in $|\Phi_0\rangle$

$$\hat{T}\,|\Phi_0\rangle = \left(\sum_{\mathbf{p}\mu,\, p<\hbar k_F} \frac{\mathbf{p}^2}{2m}\right) |\Phi_0\rangle\,. \tag{208}$$

The energy of the ground state is then obtained by taking the appropriate continuum limit discussed above

$$E_0 = \left(\sum_{\mathbf{p}\mu,\, p<\hbar k_F} \frac{\mathbf{p}^2}{2m}\right) = \frac{V}{(2\pi)^3}\sum_\mu \int d^3k \frac{\hbar^2 k^2}{2m}\theta(k_F - k) = V\frac{\nu}{(2\pi)^3}\, 4\pi\, \frac{\hbar^2}{2m}\, \frac{1}{5} k_F^5. \tag{209}$$

One may therefore write in the thermodynamic limit

$$\frac{T}{N} = \frac{V}{N}\, \frac{\nu}{2\pi^2}\, \frac{\hbar^2 k_F^5}{10m} = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m} = \frac{3}{5}\epsilon_F = \frac{3}{5}k_B T_F, \tag{210}$$

where Eq. (205) has been used and the Fermi energy and temperature have been introduced by the last equalities (k_B is Boltzmann's constant).

Various different systems qualify to be considered in terms of this simple Fermi gas model. A very important example is provided by the homogeneous electron gas which provides a first approximation to a metal or a plasma. One assumes that the positive ions yield a uniform background yielding a total system that is electrically neutral. One neglects therefore the motion of the ions which is permissible due to their much larger mass. The uniform background assumption is less reliable. Some relevant results of this model which has degeneracy factor $\nu = 2$ can be found in Ref. [1] (p. 21-31). It is customary to use the Bohr radius ($a_0 = \hbar^2/me^2 = 5.29\ 10^{-11}$m $= 0.529\mathring{A}$) to introduce a dimensionless parameter for this system given by

$$r_s = \frac{r_0}{a_0}. \tag{211}$$

A metallic element contains N_A (Avogadro's number) atoms per mole and ρ_m/A moles per cm^3, where ρ_m is the mass density (in grams per cm^3), and A is the atomic mass. If one assumes that each atom contributes Z electrons to conduction, the number of free electrons per cubic centimeter is

$$\rho = \frac{N}{V} = 6.02205 \times 10^{23} \frac{Z\rho_m}{A}. \tag{212}$$

Assuming $Z = 1$ for alkali metals yields densities (at a temperature of 5K) of 0.91 for Cs, 1.15 for Rb, 1.40 for K, and 2.65 for Na times 10^{22}/cm^3. Corresponding values of r_s are 5.62 for Cs, 5.20 for Rb, 4.86 for K, and 3.93 for Na, respectively. This translates into Fermi wavenumbers ranging from 0.65 $\mathring{A}^{-1}$ for Cs to 0.92 $\mathring{A}^{-1}$ for Na. The corresponding range of Fermi energies and temperatures is given by 1.59 eV and 1.84 $\times 10^4$ K for Cs, and 3.24 eV and 3.77 $\times 10^4$ K for Na. The electron gas model can also be used to model white dwarf stars. One typically assumes that the star contains the nuclei of He atoms (α particles) and electrons (see Refs. [32,33]). Note that for realistic conditions it is necessary to treat the electrons relativistically due to the high density of such a system.

As discussed in the previous section, one observes a density in the interior of nuclei corresponding to 0.16 nucleons per fm^3. The corresponding wavenumber according to Eq. (203) using a degeneracy factor of 4 for nuclear matter at this saturation density is therefore $k_F = 1.33$ fm^{-1}. In terms of the interparticle spacing one has $r_0 = 1.14$ fm not much larger than the charge radius of a nucleon. The interior of a neutron star (electrons can no longer prevent gravitational collapse as in white dwarfs) may be considered in first approximation as a neutron gas (degeneracy factor 2) with a wide range of densities going way beyond the saturation density of nuclear matter.

Helium atoms come in two varieties ^{3}He and ^{4}He. The lighter version is a fermion, the heavier one a boson. Systems of both types of particles exhibit spectacular quantum features at low temperature and remain liquid down to the lowest temperatures. The binding forces of these atoms (to form the liquid) are van der Waals forces which arise from the polarization induced in the electronic shells when the atoms approach each other (for a simple discussion see Ref. [26]). In addition to this attractive component which is quite weak for He atoms, there is an effective repulsion between the atoms due to the Pauli principle which is almost hard-core like. Due to their light mass there is a delicate balance between the kinetic and potential energy. At zero pressure both systems remain liquid when the temperature approaches 0 K. Both exhibit superfluidity and other remarkable properties when the temperature is lowered. For ^{3}He the observed density at zero pressure corresponds to 36.84 cm^3/mole. This translates into 0.0163 atoms/$\mathring{A}^3$ which in turn yields a wavenumber of $k_F = 0.784$ $\mathring{A}^{-1}$ using a degeneracy factor of 2 since the total spin of the atom is 1/2. At this density the binding energy per atom is 2.52 K a tiny amount compared to the atomic energy scales. For this reason it makes sense to discuss such a system from the perspective of atoms interacting through (a dominant) two-body interaction which in turn may be calculated from a more microscopic starting point and (in addition) may be obtained from information obtained at higher temperature where quantum effects no longer play a role and the kinetic and potential energy separate. The boson counterpart ^{4}He forms a system that is more bound (not surprising since there is no Pauli principle which leads to a substantial kinetic energy contribution). At a density of 0.0218 atoms/$\mathring{A}^3$ the binding energy per atom is 7.14 K. Attempts have been underway for a long time to form fully spin-polarized ^{3}He. Such a system would have a degeneracy of 1 with interesting consequences for the way in which the interaction is sampled (see below).

B. Symmetry considerations for two-particle states

It is useful to consider the consequences of the symmetry of two-particle states beyond the obvious ones discussed sofar. In particular it is frequently practical or necessary in dealing with the scattering of two particles to consider the problem in an angular momentum basis. This transformation involves going from plane waves to spherical waves and is usually treated at the sp level. The consequences of symmetry are substantial, however, and will be illustrated first for two nucleons in free space. An appropriate one-nucleon state can be labeled by momentum, spin (1/2), spin projection, isospin (1/2), and isospin projection

$$|\mathbf{p}\ 1/2\ m_s\ 1/2\ m_t\rangle = |\mathbf{p} m_s m_t\rangle\,. \tag{213}$$

Antisymmetry for two nucleons requires the two-body state to be constructed as follows

$$
\begin{aligned}
|\mathbf{p}_1 m_{s_1} m_{t_1}; \mathbf{p}_2 m_{s_2} m_{t_2}\rangle &= \frac{1}{\sqrt{2}}\{|\mathbf{p}_1 m_{s_1} m_{t_1}\rangle\,|\mathbf{p}_2 m_{s_2} m_{t_2}\rangle - |\mathbf{p}_2 m_{s_2} m_{t_2}\rangle\,|\mathbf{p}_1 m_{s_1} m_{t_1}\rangle\} \\
&= \frac{1}{\sqrt{2}}\sum_{SM_S}\sum_{TM_T}\{(1/2\ m_{s_1}\ 1/2\ m_{s_2}\ |S\ M_S)(1/2\ m_{t_1}\ 1/2\ m_{t_2}\ |T\ M_T)\,|\mathbf{p}_1\ \mathbf{p}_2\ S\ M_S\ T\ M_T\rangle \\
&- (1/2\ m_{s_2}\ 1/2\ m_{s_1}\ |S\ M_S)(1/2\ m_{t_2}\ 1/2\ m_{t_1}\ |T\ M_T)\,|\mathbf{p}_2\ \mathbf{p}_1\ S\ M_S\ T\ M_T\rangle\}\,,
\end{aligned}
\tag{214}
$$

where the spins and isospins have been coupled to total spin and isospin in the second equality. Since the dynamics is related to the relative motion of the particles, it is appropriate to switch to a basis involving the center of mass (total) and relative momentum

$$
\begin{aligned}
\mathbf{P} &= \mathbf{p}_1 + \mathbf{p}_2 \\
\mathbf{p} &= \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)\,.
\end{aligned}
\tag{215}
$$

The states in the last line of Eq. (214) then both have the same total momentum but opposite relative momentum, $\mathbf{p}$ and $-\mathbf{p}$, respectively. The transformation of the relative momentum quantum number to the basis with the magnitude of this momentum, orbital angular momentum, and its projection is given by

$$
|\mathbf{p}\rangle = \sum_{LM_L} |pLM_L\rangle\,\langle LM_L|\hat{\mathbf{p}}\rangle = \sum_{LM_L} |pLM_L\rangle\, Y^*_{LM_L}(\hat{\mathbf{p}}) \tag{216}
$$

$$
|-\mathbf{p}\rangle = \sum_{LM_L} |pLM_L\rangle\,\langle LM_L|\widehat{-\mathbf{p}}\rangle = \sum_{LM_L} |pLM_L\rangle\,(-1)^L Y^*_{LM_L}(\hat{\mathbf{p}}), \tag{217}
$$

where the following property of the spherical harmonics has been used in the last equation

$$
Y^*_{LM_L}(\widehat{-\mathbf{p}}) = Y^*_{LM_L}(\pi - \theta_p, \phi_p + \pi) = (-1)^L Y^*_{LM_L}(\hat{\mathbf{p}}). \tag{218}
$$

One may now use the symmetry properties of the Clebsch-Gordan coefficients

$$
\begin{aligned}
(1/2\ m_{s_2}\ 1/2\ m_{s_1}\ |S\ M_S) &= (-1)^{\frac{1}{2}+\frac{1}{2}-S}\ (1/2\ m_{s_1}\ 1/2\ m_{s_2}\ |S\ M_S) \\
(1/2\ m_{t_2}\ 1/2\ m_{t_1}\ |T\ M_T) &= (-1)^{\frac{1}{2}+\frac{1}{2}-T}\ (1/2\ m_{t_1}\ 1/2\ m_{t_2}\ |T\ M_T)\,,
\end{aligned}
\tag{219}
$$

to write Eq. (214) as

$$
\begin{aligned}
|\mathbf{p}_1 m_{s_1} m_{t_1}; \mathbf{p}_2 m_{s_2} m_{t_2}\rangle &= \frac{1}{\sqrt{2}}\sum_{SM_S}\sum_{TM_T}\sum_{LM_L}(1/2\ m_{s_1}\ 1/2\ m_{s_2}\ |S\ M_S)(1/2\ m_{t_1}\ 1/2\ m_{t_2}\ |T\ M_T) \\
&\times Y^*_{LM_L}(\hat{\mathbf{p}})\left[1-(-1)^{L+S+T}\right]|\mathbf{P}\ p\ LM_L SM_S\ TM_T\rangle \\
&= \frac{1}{\sqrt{2}}\sum_{SM_S}\sum_{TM_T}\sum_{LM_L}\sum_{JM_J}(1/2\ m_{s_1}\ 1/2\ m_{s_2}\ |S\ M_S)(1/2\ m_{t_1}\ 1/2\ m_{t_2}\ |T\ M_T) \\
&\times Y^*_{LM_L}(\hat{\mathbf{p}})(L\ M_L\ S\ M_S\ |J\ M_J)\left[1-(-1)^{L+S+T}\right]|\mathbf{P}\ p\ (LS)JM_J\ TM_T\rangle\,,
\end{aligned}
\tag{220}
$$

where the coupling to total angular momentum has been performed after the last equality sign. The main point of this result is the factor $\left[1-(-1)^{L+S+T}\right]$. It shows that only for $L+S+T$ odd a physical antisymmetric state can occur. Consider the possibilities of an S-wave interaction. In this case $L=0$ and therefore only two possibilities exist: either $S=0$ and $T=1$ or $S=1$ and $T=0$. The spectroscopic notation for the different channels is given by ${}^{2S+1}L_J$ where the actual values of S and J are inserted and the letter notation for L is used ($L=0$ corresponds to S, $L=1$ to P, $L=2$ to D etc.). So the two S-wave channels for nucleons are denoted by ${}^1S_0$ and ${}^3S_1$. Since the strong interaction conserves parity and is a scalar with respect to rotations generated by $\mathbf{J}$ and $\mathbf{T}$, the total angular momentum and total isospin, the couplings between different channels must keep the same J and T and can change the L-value by at most 2 (due to the fact that the particles have spin 1/2). This implies that the ${}^1S_0$ two-proton channel is uncoupled whereas the proton-neutron channel allows a coupling between the ${}^3S_1$ and ${}^3D_1$ channels. The latter coupling is realized in nature due to the presence of the so-called tensor force which is instrumental in binding the deuteron and giving it its quadrupole moment.

If we now turn to antisymmetric two-particle states for electrons or ^{3}He atoms which have spin 1/2, one can use the above results and simply remove all referral to isospin. The corresponding factor that decides which partial wave channels are physically allowed then becomes $\left[1+(-1)^{L+S}\right]$. This shows that an S-wave interaction implies a total spin of zero, whereas a P-wave requires a total spin of one, etc. Since for these systems one normally doesn't need to consider tensor forces, one may forego the coupling to total angular momentum states. In case there is only one spin projection of the species available, the consequences of the Pauli principle are even more dramatic. The Pauli factor now becomes $\left[1-(-1)^{L}\right]$ showing that there can be no S-wave interaction. Recent efforts to cool fermionic atoms to temperatures substantially below the Fermi temperature have to deal with this lack of S-wave interaction when cooling these systems in magnetic and optical traps.

In finite systems with spherical symmetry one may also consider the coupling of angular momentum states for two particles. The consquences of the Pauli principle are also striking here. One can illustrate this example by using second quantized notation. Consider two particles added to a closed-shell nucleus such as discussed in the previous section. For the moment assume that these two particles are either two protons or two neutrons and that they are added in the same sp shell characterized by sp angular momentum j. Such a state can be written as

$$a^\dagger_{jm} a^\dagger_{jm'} \left|\Phi_0\right\rangle . \tag{221}$$

It is immediately clear that a total angular momentum of $J=2j$ is not possible since this would require both particles to have the same projection of the angular momentum. The allowed total angular momentum states can be obtained by coupling the sp angular momenta and using the symmetry property of the Clebsch-Gordan coefficients and the anticommutation relation of the particle addition operators as follows

$$\begin{aligned} &\sum_{mm'} (j\ m\ j\ m'\ |J\ M)\, a^\dagger_{jm} a^\dagger_{jm'} \left|\Phi_0\right\rangle \\ &= \sum_{mm'} (j\ m'\ j\ m\ |J\ M)\, a^\dagger_{jm'} a^\dagger_{jm} \left|\Phi_0\right\rangle \\ &= \sum_{mm'} (-1)^{2j-J} (j\ m\ j\ m'\ |J\ M)\, (-1) a^\dagger_{jm} a^\dagger_{jm'} \left|\Phi_0\right\rangle \\ &= (-1)^{J} \sum_{mm'} (j\ m\ j\ m'\ |J\ M)\, a^\dagger_{jm} a^\dagger_{jm'} \left|\Phi_0\right\rangle , \end{aligned} \tag{222}$$

where a change of dummy indices was used in the first equality. The factor $(-1)^J$ ensures that only even values of J yield physical states. To include isospin one proceeds in a similar fashion and obtains

$$\begin{aligned} &\sum_{mm'} \sum_{tt'} (j\ m\ j\ m'\ |J\ M)\, (1/2\ t\ 1/2\ t'\ |T\ M_T)\, a^\dagger_{jmt} a^\dagger_{jm't'} \left|\Phi_0\right\rangle \\ &= (-1)^{J+T+1} \sum_{mm'} \sum_{tt'} (j\ m\ j\ m'\ |J\ M)\, (1/2\ t\ 1/2\ t'\ |T\ M_T)\, a^\dagger_{jmt} a^\dagger_{jm't'} \left|\Phi_0\right\rangle . \end{aligned} \tag{223}$$

In this case $J+T$ must be odd. This result is consistent with the previous one since for two protons or two neutrons the total isospin must be one. It is useful to determine the normalization of the states considered above.

Spinless bosons can also be coupled to good relative orbital angular momentum. Similar steps as used for two fermions can be followed while properly accounting for the symmetry of the two-particle state. It is not surprising that this results in states that can only have even orbital angular momentum.

C. Interactions

The main problem in dealing with a many-particle system is to properly deal with the basic interaction between the constituent particles. Before giving some explicit examples of these interactions and their matrix elements it is useful to put this discussion in a wider perspective. All interactions between the particles of the Standard Model of the Electroweak and Strong Interactions take place by an exchange mechanism between fermions which have spin 1/2. These fermions include all the quarks which come in three colors and six flavors, and all the leptons which include the electron, the muon, and the tau together with their corresponding neutrino's. In the case of these elementary particles, the exchanged particle is always a boson with spin 1 like the photon, one of the gluons, or a heavy vector boson. The best known example corresponds to the exchange of a photon between electrons. In general, interactions between particles in **any** setting can be discussed in terms of an exchange mechanism. Depending on the circumstances the "particle" may be a low-energy bosonic excitation (of any spin) of the medium which is exchanged between the fermion constituents. For example, electrons in a solid can exchange the lattice vibrations (phonons) of the core atoms. It will be useful to keep this exchange mechanism in mind even though it is not always obvious that it is at work. An example of such a case is provided by the instantaneous Coulomb repulsion between two electrons which does originate from the one-photon exchange mechanism (see section #VI).

Apart from their obvious thermodynamic relevance, one cannot overemphasize the importance of the low-energy excitations of a many-particle system. Their excitation energy provides a new energy scale which is not present in the vacuum. This can be illustrated at several levels by considering nuclei. The nucleons making up the nucleus are themselves composite objects made up of quarks in such a way that no explicit color is present. A nucleon can thus be considered as the lowest bound state of three quarks with total angular momentum and isospin 1/2. Experimentally, it has been impossible to isolate individual quarks up to now and one can therefore say that the energy scale associated with generating individual quarks is infinite for all practical purposes, although QCD (Quantum Chromo Dynamics) in the high-energy and experimentally accessible domain describes a weakly interacting system of massless quarks predominantly interacting by the exchange of single massless gluons. In the low-energy domain, where non-perturbative effects and confinement dominate, the lowest excited states of QCD are found at excitation energies of the order of hundreds of MeV. A particularly important example is the Δ-isobar at 1232 MeV which has angular momentum and isospin 3/2. The energy difference between the nucleon and the Δ provides a new energy scale. In addition there are bosonic excitation modes of QCD which can be interpreted in terms of quark-antiquark states. The lowest-energy state is the pion with angular momentum 0 and isospin 1 which also has opposite parity from the nucleon and the Δ. The energy of the pion is about 140 MeV. One consequence of the low energy of the pion and the experimental result that it couples strongly to the nucleon is that the exchange mechanism discussed above is very important. This means that part of the interaction between two nucleons is represented by the exchange of individual pions. Since pions are the lowest-mass mesons, their exchange provides that part of the interaction which has the longest range illustrating the connection between the mass (energy) of the exchanged particle and the range of the interaction. The idea of a meson-exchange mechanism to describe the strong interaction dates back to Yukawa [34]. Exchange mechanisms of higher-energy mesons with other quantum numbers can then be used to describe the interaction between two nucleons at shorter range. The energy scale which involves the explicit excitation of other QCD states therefore starts at 140 MeV with an additional important state (the Δ) at 300 MeV.

This first example demonstrates that the elementary excitations of the QCD field theory are subject to a different energy scale compared to the non-interacting free field theory in which the quarks and gluons have no mass. In the interacting theory the colorless bound states dominate at low energy and explicit single quark and gluon degrees of freedom with color are effectively at infinite excitation energy. Depending on the objectives one must therefore make a choice which are the relevant degrees of freedom which determine the properties of the system under study: In the case of one nucleon one may attempt a solution of QCD on the lattice, in the case of many nucleons it is usually more fruitful to start from nucleons which interact by means of the above outlined meson-exchange mechanism using input from experimental data. This approach certainly makes sense from the point of view of perturbation theory. Indeed, from overwhelming experimental evidence it is clear that nucleons maintain much of their identity when they are brought together with other nucleons.

In general, one can say that the experimentally observed low-lying excitations of a system themselves play an important role in understanding the physics of the many-particle system. Sometimes these excitations are referred to as elementary modes of excitation or quasiparticles. To understand the physics it may be important to treat the interaction between these modes. In liquid ^{4}He one finds single-particle-like excitations, which are usually called quasi-particles, but also bosonic collective excitations like phonons and rotons. One level of understanding is achieved by describing the liquid in terms of atoms interacting with each other, another level is achieved by working with phonons and rotons [21]. Whereas the latter mode of description has certainly not yet been completely successful in bringing about a microscopic understanding of rotons, the former description mode is restricted by its inherent phenomenological character. Indeed, even if one can numerically calculate certain properties of the system microscopically, this does not imply that one understands the physics very well at the same time.

In the case of a nucleus one cannot even think of abandoning the nucleon description before one can solve QCD with an accuracy better than the lowest energy scale that is relevant for the system under study. In a nucleus the lowest excitation modes have energies of the order of MeV's in light nuclei, but in heavy nuclei the lowest excited state (of boson character) may be at about 50 keV. Here we are again dealing with other energy scales that are introduced because nucleons are brought together in a nucleus. The nucleons, because they experience an overall attractive interaction with each other, clump together to form a self-bound system with a size dictated by the effective interactions (interactions in the nuclear medium) between the nucleons. This size in turn introduces a new energy scale. One can understand this from a sp point of view in which the nucleons find themselves bound in the average attractive field of the other nucleons. This potential is obviously of the size of the nucleus and as a consequence the new energy scale related to the sp energies of this potential is introduced. Since now and in the foreseeable future QCD will not be understood with an accuracy which is relevant for understanding the physics of the nucleus at energies ranging from the lowest excited states to a few hundred MeV, it is more profitable to use the elementary mode of excitation description which can be applied to any many-particle system.

In a similar vein it is hard to conceive that one would be interested in understanding the Helium liquids in terms of a hamiltonian at the level of the Coulomb interactions between the electrons, the alpha particles, and the electrons and the alpha particles (or ^{3}He nuclei). Instead, many-particle theory attempts to explain the properties associated with the relevant energy scales of the liquid, which is in degrees Kelvin (again a new energy scale associated with the many-particle nature of the system), in terms of Helium atoms which experience an "effective" interaction characterizing the behavior of isolated atoms in free space. This interaction takes into account both the polarization effect of the electron cloud of one atom on the cloud of another, representing the long-range part of the interaction, as well as simulates the effect of the Pauli principle between the electrons in the different atoms. The latter effect becomes very important when the atoms are brought close together and results in a strongly repulsive effective interaction between the atoms at short distances. This Lennard-Jones type interaction is therefore used to simulate effects that are associated with degrees of freedom that are important at higher energy scales. It should be noted that electronic excitation energies are typically 4-5 orders of magnitude larger than the relevant energy scale in the liquid.

A similar simplification must be made in the case of electrons in a solid. The original hamiltonian describing the Coulomb interaction between nuclei, nuclei and electrons, and electrons and electrons is too general to provide a realistic starting point for the description of the solid state. In this context it is good to repeat some basic questions associated with the physics of solids which were formulated by Anderson [4,13] (without attempting any answers at this time) :

1. Why is a solid?
2. How does one describe a solid from a truly fundamental point of view in which atomic nuclei as well as the electrons are treated truly quantummechanically?
3. How and why does a solid hold itself together?

D. Explicit result for two-body interactions

After this general perspective it is useful to discuss some explicit results for the two-body matrix elements of certain useful interactions. These matrix elements represent the *c*-number content of the Fock-space two-body operator discussed section #II. Very often an interaction was developed theoretically or phenomenologically by demanding a description of certain scattering properties in free space, in the coordinate representation. In many cases no explicit spin (or isospin) dependence needs to be considered. In addition such an interaction will only depend on the relative coordinate between the particles. In the case of spherical symmetry this dependence is further reduced to only the magnitude of this relative distance. For such an interaction two-body matrix elements considered in coordinate space yield the following results (suppressing discrete quantum numbers)

$$\begin{aligned} \langle \mathbf{r}_1\mathbf{r}_2 | V(r_{op}) | \mathbf{r}_3\mathbf{r}_4 \rangle &= \langle \mathbf{R}\mathbf{r} | V(r_{op}) | \mathbf{R}'\mathbf{r}' \rangle \\ &= \delta(\mathbf{R}-\mathbf{R}') \langle \mathbf{r} | V(r_{op}) | \mathbf{r}' \rangle \\ &= \delta(\mathbf{R}-\mathbf{R}')\delta(\mathbf{r}-\mathbf{r}')V(r), \end{aligned} \tag{224}$$

where a transformation to center-of-mass and relative coordinates has been used

$$\begin{aligned} \mathbf{R} &= \frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2) \\ \mathbf{r} &= \mathbf{r}_1-\mathbf{r}_2, \end{aligned} \tag{225}$$

and similarly for the primed coordinates. An interaction with such matrix elements in coordinate space is also called a local interaction. Not all relevant interactions are local. For a Coulomb interaction between charges q_1e and q_2e one simply has

$$V_C(r) = \frac{q_1q_2e^2}{r}. \tag{226}$$

Another useful interaction is the socalled Yukawa interaction given by

$$V_Y(r) = V_0\frac{e^{-\mu r}}{\mu r}. \tag{227}$$

Matrixelements of such interactions in momentum space are needed when one considers an infinite homogeneous system. Using the transformation to total and relative momenta considered before, one obtains (using the wave functions introduced at the beginning of this section)

$$\begin{aligned}\langle \mathbf{p}_1\mathbf{p}_2|\, V(r)\, |\mathbf{p}_3\mathbf{p}_4\rangle &= \langle \mathbf{P}\mathbf{p}|\, V(r)\, |\mathbf{P}'\mathbf{p}'\rangle \\ &= \delta_{\mathbf{P},\mathbf{P}'} \langle \mathbf{p}|\, V(r)\, |\mathbf{p}'\rangle .\end{aligned} \tag{228}$$

One may also use wave vectors for the matrix element for the relative motion. Using this basis one needs to calculate

$$\langle \mathbf{k}|\, V(r)\, |\mathbf{k}'\rangle = \frac{1}{V}\int d^3r\; e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}} V(r). \tag{229}$$

This matrix element can be manipulated further by using the standard expansion

$$e^{i\mathbf{q}\cdot\mathbf{r}} = 4\pi \sum_{\ell m} i^\ell Y^*_{\ell m}(\hat{\mathbf{r}}) Y_{\ell m}(\hat{\mathbf{q}}) j_\ell(qr), \tag{230}$$

where j_ℓ is the spherical Bessel function. Inserting this result in Eq. (229) and performing the angular integration one obtains

$$\langle \mathbf{k}|\, V(r)\, |\mathbf{k}'\rangle = \frac{4\pi}{V}\int dr\; r^2\; j_0(qr) V(r), \tag{231}$$

with $q = |\mathbf{k}-\mathbf{k}'|$. For the case of the Yukawa interaction of Eq. (227) this last integral can *e.g.* be found in Ref. [35] with the result

$$\langle \mathbf{k}|\, V_Y(r)\, |\mathbf{k}'\rangle = \frac{4\pi}{V}\frac{V_0}{\mu}\frac{1}{\mu^2 + (\mathbf{k}'-\mathbf{k})^2}. \tag{232}$$

This result can also be used to obtain the matrix element of the Coulomb interaction

$$\langle \mathbf{k}|\, V_C(r)\, |\mathbf{k}'\rangle = \frac{4\pi}{V}\frac{q_1 q_2 e^2}{(\mathbf{k}'-\mathbf{k})^2}, \tag{233}$$

where the case $\mathbf{k} = \mathbf{k}'$ requires special consideration but can usually be omitted on account of cancellations as in the case of the homogeneous electron gas [1,18–20].

Another type of interaction that may be useful to consider is of the following form

$$V(r) = A\; e^{-\alpha r}. \tag{234}$$

Such an interaction may be used to describe the short-range part of the atom-atom repulsion [36]. Noting that the relevant momentum space matrix element only depends on the magnitude of the transferred momentum $q = |\mathbf{q}| = |\mathbf{k}'-\mathbf{k}|$ between the particles in the case of a local interaction, one can show that for the interaction of Eq. (234) one obtains

$$\langle \mathbf{k}|\, V(r)\, |\mathbf{k}'\rangle = V(q) = -\frac{4\pi A}{V}\frac{d}{d\alpha}\;\frac{1}{\alpha^2+q^2}. \tag{235}$$

This form is useful also when one is interested in obtaining matrix elements in a partial-wave basis.

In a partial-wave basis one is looking for matrix elements of the form

$$\langle kLM|\, V\, |k'L'M'\rangle = \int d\hat{\mathbf{k}}\; \langle LM|\hat{\mathbf{k}}\rangle \int d\hat{\mathbf{k}}'\; \langle \hat{\mathbf{k}}'|L'M'\rangle \langle \mathbf{k}|\, V(r)\, |\mathbf{k}'\rangle . \tag{236}$$

In the case of a Yukawa interaction one may now proceed by rewriting Eq. (232) in the following form

$$\langle \mathbf{k}|\, V_Y(r)\, |\mathbf{k}'\rangle = \frac{4\pi}{V}\frac{V_0}{\mu}\frac{1}{2kk'}\frac{1}{\frac{\mu^2+k^2+k'^2}{2kk'} - \cos\theta_{kk'}}. \tag{237}$$

This last fraction may be expanded using the following relation involving Legendre function Q_ℓ and Legendre polynomials P_ℓ

$$\frac{1}{\frac{\mu^2+k^2+k'^2}{2kk'} - \cos\theta_{kk'}} = \sum_{\ell=0}^{\infty} (2\ell+1)\ Q_\ell\left(\frac{\mu^2+k^2+k'^2}{2kk'}\right) P_\ell(\cos\theta_{kk'})$$
$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} 4\pi\ Q_\ell\left(\frac{\mu^2+k^2+k'^2}{2kk'}\right) Y^*_{\ell m}(\hat{\mathbf{k}}) Y_{\ell m}(\hat{\mathbf{k}'}). \tag{238}$$

In the last equality the addition theorem for spherical harmonics was used. Note that the argument of the Legendre function must be larger than 1 while the argument of the Legendre polynomial must be less than 1. Inserting these results into Eq. (236) one finally obtains after performing the angular integrations

$$\langle kLM|\,V\,|k'L'M'\rangle = \delta_{LL'}\ \delta_{MM'}\frac{4\pi}{V}\frac{V_0}{\mu}\frac{1}{2kk'}\ 4\pi\ Q_\ell\left(\frac{\mu^2+k^2+k'^2}{2kk'}\right). \tag{239}$$

The first three Legendre functions are given by

$$Q_0(z) = \frac{1}{2}\ \ln\left(\frac{z+1}{z-1}\right)$$
$$Q_1(z) = \frac{z}{2}\ \ln\left(\frac{z+1}{z-1}\right) - 1$$
$$Q_2(z) = \frac{3z^2-1}{4}\ \ln\left(\frac{z+1}{z-1}\right) - \frac{3}{2}z. \tag{240}$$

The most complicated two-body interaction that needs to be considered is probably the one between nucleons. Some of this complication is associated with the operator structure that is required to describe this interaction. In addition to the usual spin/isospin independent term discussed above, one has to include spin and isospin dependent operators each associated with their own characteristic spatial dependence. Most likely it will be of some relevance to consider the interaction between nucleons to be nonlocal as well. It is possible to give an accurate account of the scattering of two nucleons up to the threshold of pion production by considering an interaction of the following form [37]:

$$v^{14}(1,2) = \sum_{p=1,14} \left[v^p_\pi(r_{12}) + v^p_I(r_{12}) + v^p_S(r_{12})\right]\ O^p_{12}. \tag{241}$$

The local radial dependence ($r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$) is governed by a long-range pion-exchange term, v^p_π, an intermediate-range part v^p_I, and a short-range contribution, v^p_S. Fourteen operators O^p_{12} need to be considered

$$\begin{array}{ccccc} 1 & \tau_1\cdot\tau_2 & \sigma_1\cdot\sigma_2 & \sigma_1\cdot\sigma_2\ \tau_1\cdot\tau_2 & S_{12} \\ S_{12}\ \tau_1\cdot\tau_2 & \mathbf{L}\cdot\mathbf{S} & \mathbf{L}\cdot\mathbf{S}\ \tau_1\cdot\tau_2 & \mathbf{L}^2 & \mathbf{L}^2\ \tau_1\cdot\tau_2 \\ \mathbf{L}^2\ \sigma_1\cdot\sigma_2 & \mathbf{L}^2\ \sigma_1\cdot\sigma_2\ \tau_1\cdot\tau_2 & (\mathbf{L}\cdot\mathbf{S})^2 & (\mathbf{L}\cdot\mathbf{S})^2\ \tau_1\cdot\tau_2 & \end{array} \tag{242}$$

This set of operators contains the usual Pauli spin and isospin matrices, the tensor operator

$$S_{12} = 3\,(\sigma_1\cdot\hat{\mathbf{r}}_{12})\,(\sigma_2\cdot\hat{\mathbf{r}}_{12}) - \sigma_1\cdot\sigma_2, \tag{243}$$

the relative orbital angular momentum $\mathbf{L}$, and the total spin $\mathbf{S}$ of the pair. Employing a partial-wave basis one can use standard angular momentum techniques to determine the matrix elements of these operators [26]. Clearly it is necessary to include a coupling to total angular momentum to keep such calculations manageable.