Chapter 1

Identical particles

In this chapter some basic notions related to identical particles are developed. In Sec. 1.1 some simple estimates are discussed which help to identify under which conditions one expects quantum phenomena related to identical particles to occur. Sec. 1.2 is devoted to a brief discussion of the theoretical and experimental background which suggest that only certain many-particle states are realized in nature. After a brief review of some relevant notation for one-particle quantum mechanics, the case of two identical particles is discussed in Sec. 1.3. In Sec. 1.4 some illustrative examples are discussed which clarify the experimental consequences related to identical particles. Finally, in Sec. 1.5 the construction and properties of states with N identical fermions or bosons is developed.

1.1 Some simple considerations

In a quantum many-body system particles of the same species are completely indistinguishable. Moreover, even in the absence of mutual interactions they still have a profound influence on each other, since the number of ways in which the same quantum state can be occupied by two or more particles is severely restricted. This is a consequence of the so-called spin-statistics theorem, which is further discussed in the next section. One may expect that such effects do not play a role when the number of possible quantum states is much larger than the number of particles, since it is unlikely that two particles would then occupy the same quantum state. This argument provides a rough-and-ready estimate of the conditions under which quantum phenomena related to identical particles are important.

Consider the energy levels for a particle of mass m in a box with volume

$V = L^3$

$$\varepsilon_{n_x,n_y,n_z} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2), \tag{1.1}$$

where h is Planck's constant and the n_i can be any nonzero positive integer. The number of states $\Omega(E)$ below an energy E is given by

$$\Omega(E) = \frac{\pi}{6}\left(\frac{8mL^2E}{h^2}\right)^{3/2} = \frac{\pi}{6}\left(\frac{8mE}{h^2}\right)^{3/2} V, \tag{1.2}$$

where E is assumed to be large enough so that $\Omega(E)$ is essentially a continuous function of energy (see *e.g.* [McQuarrie (1976)]). If we take the average energy of a particle to be $E = 3/2k_BT$, where k_B is Boltzmann's constant and T the temperature in kelvin, one can check that in a box with $L = 10$ cm and at $T = 300$ K the number of states Ω for an atom with mass $m = 10^{-25}$ kg is about 10^{30}. This is much larger than the number of atoms N in the box under normal conditions of temperature and pressure. Generalizing this argument, while requiring $N \ll \Omega$, one does not expect quantum indistinguishability effects to play a role when

$$1 \gg Q \equiv \frac{N}{\Omega} = \frac{6}{\pi}\rho\left(\frac{h^2}{12mk_BT}\right)^{3/2}, \tag{1.3}$$

where $\rho = N/V$ is the particle density and Eq. (1.2) was used with E replaced by $3/2k_BT$. Large particle mass, high temperature and low density favor this condition. Small mass, low temperature, and high density on the other hand favor the appearance of quantum effects associated with identical particles.

The dimensionless quantity Q is listed in Table 1.1 for a number of many-body systems. For atoms and molecules one only expects quantum effects for the very light ones, at low temperatures. For electrons in metals, however, the condition (1.3) is already dramatically violated at 273 K. In a white dwarf star the temperature is much higher, but a quantum treatment of the electrons is still mandatory because of the extreme density. For the protons and neutrons in nuclei, at a typical nuclear energy scale of about 1 MeV or 10^{10} K the condition (1.3) is also severely violated, and the same holds true for the neutrons in in a neutron star at $T = 10^8$ K (which is rather cool according to nuclear standards). A vapor of alkali atoms (rubidium), even when very dilute, nevertheless shows a spectacular quantum effect when cooled down to extremely low temperatures: the formation of

Table 1.1 Q-parameter for different systems

System	T (K)	Density (m^{-3})	Mass (u)	Q
He (l)	4.2	1.9×10^{28}	4.0	1.1
He (g)	4.2	2.5×10^{27}	4.0	1.4×10^{-1}
He (g)	273	2.7×10^{25}	4.0	2.9×10^{-6}
Ne (l)	27.1	3.6×10^{28}	20.2	1.1×10^{-2}
Ne (g)	273	2.7×10^{25}	20.2	2.5×10^{-7}
e^- Na metal	273	2.5×10^{28}	5.5×10^{-4}	1.7×10^{3}
e^- Al metal	273	1.8×10^{29}	5.5×10^{-4}	1.2×10^{4}
e^- white dwarfs	10^7	10^{36}	5.5×10^{-4}	8.5×10^{3}
p,n nuclear matter	10^{10}	1.7×10^{44}	1.0	6.5×10^{2}
n neutron star	10^8	4.0×10^{44}	1.0	1.5×10^{6}
^{87}Rb condensate	10^{-7}	10^{19}	87	1.5

The dimensionless quantity $Q = 6/\pi\rho(h^2/12mk_BT)^{3/2}$ for a number of many-body systems, using representative values of densities and temperatures. The mass of the particles is given in atomic mass units (u). Helium and neon are considered at atmospheric pressure, with the liquid phase at boiling point. Electrons in the metals sodium and aluminum can be compared to electrons in white dwarf stars. Protons and neutrons at saturation density of nuclear matter (the density observed in the interior of heavy nuclei) are considered as well as neutrons in the interior of neutron stars. The last entry is the Bose-Einstein condensate of a dilute vapor of ^{87}Ru atoms, magnetically trapped and cooled to ca. 100 nK.

a so-called Bose-Einstein condensate, which was recently achieved experimentally [Wiemand and Cornell (1995)].

Similar estimates for the importance of quantum effects are obtained by considering the thermal wavelength of a particle which is given by

$$\lambda_T = \left[\frac{h^2}{2\pi m k_B T}\right]^{1/2} \tag{1.4}$$

for a particle with mass m and energy k_BT. When λ_T^3 becomes comparable with the volume per particle (V/N) one expects the identity of particles to play a significant role.

1.2 Bosons and fermions

Spin and statistics are related at the level of quantum field theory [Streater and Wightman (2000)]. The Dirac equation for a spin-1/2 fermion cannot be quantized without insisting that the field operators obey anticommutation relations. These relations, in turn, lead to Fermi-Dirac statistics represented by the Pauli exclusion principle for fermions. Fermions comprise all fundamental particles with half-integer intrinsic spin. Similarly, the quantization of Maxwell's equations without sources and currents, is only possible when commutation relations between the field operators are imposed leading to Bose-Einstein statistics. Bosons can be identified by integer intrinsic spin appropriate for fundamental particles like photons and gluons. Pais has given a historical perspective on the development of quantum statistics [Pais (1986)].

Many interesting many-particle systems contain fermions as their basic constituents. Without recourse to quantum field theory (QFT) one can treat the consequences of the identity of spin-1/2 particles as a result which is based on experimental observation. Indeed, this is how Pauli came to formulate his famous principle [Pauli (1925)]. By analyzing experimental Zeeman spectra of atoms, he concluded that electrons in the atom could not occupy the same single-particle (sp) quantum state. In order to deal with this observation derived from experiment, it is necessary to postulate that quantum states which describe N identical fermions must be antisymmetrical upon interchange of any two of these particles. A similar postulate requiring symmetric states upon interchange pertains for quantum states of N identical bosons. Also here one can invoke experimental evidence to insist on symmetric states in order to account for Planck's radiation law [Pais (1986)]. It appears that only symmetric or antisymmetric many-particle states are encountered in nature.

1.3 Antisymmetric and symmetric two-particle states

To implement these postulates and study their consequences, it is useful to repeat a few simple relations of sp quantum mechanics that also play an important role in many-particle quantum physics. Useful texts on Quantum Mechanics where this background material can be found are [Sakurai (1994); Messiah (1999)]. A sp state is denoted in Dirac notation by a ket $|\alpha\rangle$, where α represents a complete set of sp quantum numbers. For a fermion, α can represent the position quantum numbers, $\boldsymbol{r}$, its total spin s (which is usually omitted), and m_s the component of its spin along the z-axis. For a spinless boson the position quantum numbers, $\boldsymbol{r}$, may be chosen. Many other possible complete sets of quantum numbers can be considered. The most relevant choice usually depends on the specific problem considered and this holds also true in a many-particle setting. This choice will be further discussed when the independent particle model is introduced in Ch. 3.

To keep the presentation general, the notation $|\alpha\rangle$ will be used but when discussing specific examples, relevant choices of sp quantum numbers will be employed.

The sp states form a complete set with respect to some complete set of commuting observables like the position operator, the total spin, and its third component. They are normalized such that

$$\langle\alpha|\beta\rangle = \delta_{\alpha,\beta} \tag{1.5}$$

where the Kronecker symbol is used to include the possibility of δ-function normalization for continuous quantum numbers. For eigenstates of the position operator one has for example

$$\langle \boldsymbol{r}, m_s | \boldsymbol{r}', m_s' \rangle = \delta(\boldsymbol{r} - \boldsymbol{r}')\delta_{m_s,m_s'} \tag{1.6}$$

for a spin-1/2 fermion. For a spinless boson

$$\langle \boldsymbol{r} | \boldsymbol{r}' \rangle = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{1.7}$$

is appropriate in this representation. The completeness of the sp states makes it possible to write the unit operator as

$$\sum_\alpha |\alpha\rangle\langle\alpha| = 1. \tag{1.8}$$

In the case of continuous quantum numbers one must use an integration instead of a summation or a combination of both in the case of a mixed spectrum.

The complex vector space relevant for N particles can be constructed as the direct product space of the corresponding sp spaces [Messiah (1999)]. Complete sets of states for N particles can be obtained by forming the appropriate product states. The essential ideas can already be elucidated by considering two particles. In this case the notation (note the rounded bracket in the ket)

$$|\alpha_1\alpha_2) = |\alpha_1\rangle\,|\alpha_2\rangle \tag{1.9}$$

is introduced. The first ket on the right-hand side of this equation refers to particle 1 and the second to particle 2. Such product states obey the following normalization condition

$$(\alpha_1\alpha_2|\alpha_1'\alpha_2') = \delta_{\alpha_1\alpha_1'}\delta_{\alpha_2\alpha_2'} \tag{1.10}$$

and completeness relation

$$\sum_{\alpha_1 \alpha_2} |\alpha_1 \alpha_2)(\alpha_1 \alpha_2| = 1. \tag{1.11}$$

While these product states are sufficient for two non-identical particles they do not incorporate the correct symmetry required to describe identical bosons or fermions. Indeed, for α_1 and α_2 different one has

$$|\alpha_2 \alpha_1) \neq |\alpha_1 \alpha_2). \tag{1.12}$$

This represents a difficulty when one performs a measurement on this system when the two particles are identical. If one obtains α_1 for one particle and α_2 for the other, one does not know which of the states in Eq. (1.12) represents the two particles. In fact, the two particles could as well be described by

$$c_1 |\alpha_1 \alpha_2) + c_2 |\alpha_2 \alpha_1) \tag{1.13}$$

which leads to an identical set of eigenvalues when a measurement is performed. This degeneracy is known as the exchange degeneracy. This exchange degeneracy presents a difficulty because a specification of the eigenvalues of a complete set of observables does not uniquely determine the state as one expects from the general postulates of quantum mechanics [Dirac (1958)].

To display the way in which the antisymmetrization or symmetrization postulates avoid this difficulty, it is convenient to introduce permutation operators. One defines the permutation operator P_{12} by

$$P_{12} |\alpha_1 \alpha_2) = |\alpha_2 \alpha_1). \tag{1.14}$$

While introduced as interchanging the quantum numbers of the particles this operator can also be viewed as effectively interchanging the particles. Clearly,

$$P_{12} = P_{21} \text{ and } P_{12}^2 = 1. \tag{1.15}$$

Consider the Hamiltonian of two identical particles :

$$H = \frac{\boldsymbol{p}_1^2}{2m} + \frac{\boldsymbol{p}_2^2}{2m} + V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|). \tag{1.16}$$

The observables, like position and momentum, must appear symmetrically in the Hamiltonian, as in the classical case. To study the action of P_{12}, consider an operator A_1 acting on particle 1

$$A_1|\alpha_1\alpha_2) = a_1|\alpha_1\alpha_2) \tag{1.17}$$

where a_1 is an eigenvalue of A_1 contained in the set of quantum numbers α_1. Similarly, an identical operator A_2 acting on particle 2 will give

$$A_2|\alpha_1\alpha_2) = a_2|\alpha_1\alpha_2). \tag{1.18}$$

Consider

$$\begin{aligned} P_{12}A_1|\alpha_1\alpha_2) &= a_1 P_{12}|\alpha_1\alpha_2) \\ &= a_1|\alpha_2\alpha_1) \\ &= A_2|\alpha_2\alpha_1) \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} P_{12}A_1|\alpha_1\alpha_2) &= P_{12}A_1P_{12}^{-1}P_{12}|\alpha_1\alpha_2) \\ &= P_{12}A_1P_{12}^{-1}|\alpha_2\alpha_1). \end{aligned} \tag{1.20}$$

From these two results one deduces that

$$P_{12}A_1P_{12}^{-1} = \ A_2 \tag{1.21}$$

since Eqs. (1.19) and (1.20) hold for any state $|\alpha_1\alpha_2)$. As a result one has

$$P_{12}HP_{12}^{-1} = \ H \tag{1.22}$$

or

$$[P_{12}, H] = \ 0 \tag{1.23}$$

implying that both operators can be diagonal simultaneously. In the case that $\alpha_1 \neq \alpha_2$, the normalized eigenkets of P_{12} are:

$$|\alpha_1\alpha_2\rangle_+ = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2) + \ |\alpha_2\alpha_1)\} \tag{1.24}$$

and

$$|\alpha_1\alpha_2\rangle_- = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2) - \ |\alpha_2\alpha_1)\}, \tag{1.25}$$

with eigenvalues +1 and -1, respectively. One can define the symmetrizer

$$S_{12} = \frac{1}{2}(1 + P_{12}) \tag{1.26}$$

and antisymmetrizer

$$A_{12} = \frac{1}{2}(1 - P_{12}) \tag{1.27}$$

which applied to any linear combination of $|\alpha_1\alpha_2)$ and $|\alpha_2\alpha_1)$ will automatically generate the symmetric or antisymmetric state, respectively. In the case of identical fermions, the Pauli exclusion principle results from the requirement that an N-particle state must be antisymmetrical upon interchange of any two particles. In the case of two particles this implies that the relevant state is the antisymmetrical one (leaving out the $-$ subscript):

$$|\alpha_1\alpha_2\rangle = \frac{1}{\sqrt{2}}\left\{|\alpha_1\alpha_2) - |\alpha_2\alpha_1)\right\}. \tag{1.28}$$

This state vanishes when $\alpha_1 = \alpha_2$ incorporating Pauli's principle. The symmetric state for two bosons (Eq. (1.24)) is not yet properly normalized when $\alpha_1 = \alpha_2$ demonstrating the possibility that bosons can occupy the same sp quantum state. The properly normalized two-boson state is given by

$$|\alpha_1\alpha_2\rangle_S = \left[\frac{1}{2n_{\alpha_1}!n_{\alpha_2}!}\right]^{1/2}\left\{|\alpha_1\alpha_2) + |\alpha_2\alpha_1)\right\}, \tag{1.29}$$

where n_α denotes the number of particles in sp state α. Obviously

$$\sum_\alpha n_\alpha = 2 \tag{1.30}$$

in this case. From now on, the states for more than one particle which have angular brackets will denote the antisymmetric or symmetric states. It should also be noted that as required for fermions

$$|\alpha_2\alpha_1\rangle = -|\alpha_1\alpha_2\rangle \tag{1.31}$$

and both kets therefore represent the same physical state. Only one of these states should be counted when the completeness relation for two identical fermions is considered. In practice this can be accomplished by ordering the sp quantum numbers. Suppose one has a set of sp states labeled by discrete quantum numbers $|1\rangle, |2\rangle, |3\rangle, ...$ etc. For two particles the completeness relation in terms of antisymmetric states then reads *e.g.*

$$\sum_{i<j} |ij\rangle\langle ij| = 1, \tag{1.32}$$

although one can use an unrestricted sum as well if one corrects for the number of equivalent states

$$\frac{1}{2!}\sum_{ij} |ij\rangle \langle ij| = 1. \tag{1.33}$$

For bosons the equivalent relations to Eqs. (1.32) and (1.33) are given by

$$\sum_{i\leq j} |ij\rangle \langle ij| = 1, \tag{1.34}$$

and

$$\sum_{ij} \frac{n_i! n_j!}{2!} |ij\rangle \langle ij| = 1. \tag{1.35}$$

1.4 Some experimental consequences related to identical particles

Scattering experiments represent an ideal tool to illustrate the consequences of dealing with identical particles. In the case of two particles that have identical mass and charge but can be distinguished in some other way, say their color being red or blue, a scattering experiment performed in the center of mass of these particles can have two distinguishable outcomes for the same scattering angle. If the red particle approaches in the z-direction and detectors than can distinguish red and blue are located in the direction θ (detector D_1) and $\pi - \theta$ (detector D_2) with the z-axis, the (quantummechanical) cross section for the red particle in D_1 and the blue particle in D_2 is given by

$$\frac{d\sigma}{d\Omega}(red\ D_1, blue\ D_2) = |f(\theta)|^2 , \tag{1.36}$$

where $f(\theta)$ is the scattering amplitude. The cross section for the red particle in D_2 and the blue particle in D_1 is given by

$$\frac{d\sigma}{d\Omega}(red\ D_2, blue\ D_1) = |f(\pi - \theta)|^2 . \tag{1.37}$$

If the detectors are colorblind one cannot distinguish between these processes and the cross section for a count in D_1 becomes the sum of the two probabilities

$$\frac{d\sigma}{d\Omega}(particle\ in\ D_1) = |f(\theta)|^2 + |f(\pi - \theta)|^2 . \tag{1.38}$$

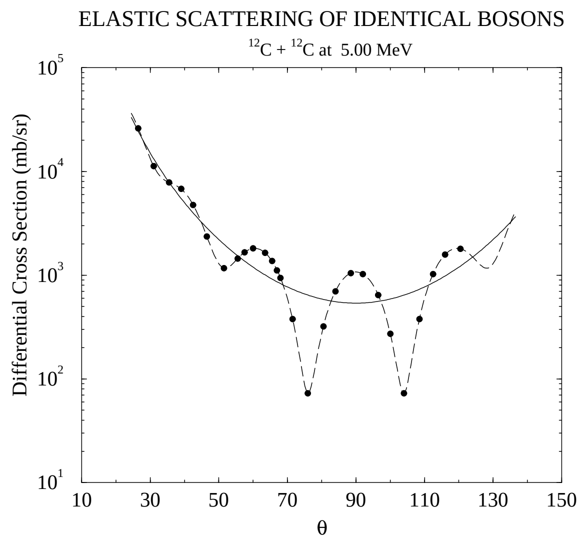

Fig. 1.1 Differential cross section of ^{12}C on ^{12}C scattering at 5.0 MeV as a function of the scattering angle θ in the center-of-mass. The full curve is obtained from the Coulomb scattering amplitude by using Eq. (1.38) while the dashed line employs the correct expression (1.39) for identical bosons. The experimental data are taken from [Bromley *et al.* (1961)].

With identical bosons both processes cannot even in principle be distinguished. This implies that the probability amplitudes must be added (the wave function of the pair must also be symmetrical) before squaring to obtain the cross section which therefore reads

$$\frac{d\sigma}{d\Omega}(bosons) = \; |f(\theta) + f(\pi - \theta)|^2 \,, \tag{1.39}$$

and now includes an interference term. The result of the interference is that at $\theta = \pi/2$ the cross section for bosons is twice that for distinguishable particles (but colorblind detectors). This prediction is confirmed by experiment as shown in Fig. 1.1. In this figure the differential cross section for the scattering of two identical ^{12}C nuclei at low energy (5 MeV) is plotted as a function of the center-of-mass scattering angle. The full line employs the Coulomb scattering amplitude [Sakurai (1994)] according to Eq. (1.38) whereas the dashed line employs Eq. (1.39). The comparison with the data [Bromley *et al.* (1961)] unambiguously points to the identical boson-boson cross section as giving the correct description. In the case of identical fermions one only obtains the interference when both particles have identical spin quantum numbers. The corresponding cross section is given by

$$\frac{d\sigma}{d\Omega}(fermions) = \left|f(\theta) - f(\pi - \theta)\right|^2 . \tag{1.40}$$

In this case no particles will be detected at all at $\theta = \pi/2$! This type of experiment does, however, require the beam and target spins to be polarized in the same direction.

It should be observed that in Fig. 1.1 the differential cross section for the scattering of composite particles is shown. This example demonstrates that it is necessary to consider these carbon nuclei as identical bosons at least at those energies where no internal excitation of one or both nuclei can occur. The critical ingredient that decides on the identity of this nucleus is the total number of fermions present, 12 in this case. For an even number of constituent fermions the composite particle behaves as a boson while it acts as a fermion when this number is odd. Examples for atoms are the ^{3}He and ^{4}He isotopes. In each case the number of electrons and protons is 2. Since ^{3}He has only one neutron, its total number of fermions is odd and a collection of these atoms will act as identical fermions. The two neutrons in the ^{4}He nucleus are responsible for the boson character of these atoms. The same reasoning demonstrates that the ^{87}Ru atom represents a boson.

1.5 Antisymmetric and symmetric many-particle states

In dealing with N particles one can proceed similarly as in the case of two particles. Product states are denoted by

$$|\alpha_1\alpha_2...\alpha_N) = |\alpha_1\rangle\,|\alpha_2\rangle\,...\,|\alpha_N\rangle \tag{1.41}$$

with orthogonality in the form

$$\begin{aligned}(\alpha_1\alpha_2...\alpha_N|\alpha_1'\alpha_2'...\alpha_N') &= \langle\alpha_1|\alpha_1'\rangle\langle\alpha_2|\alpha_2'\rangle...\langle\alpha_N|\alpha_N'\rangle \\ &= \delta_{\alpha_1,\alpha_1'}\delta_{\alpha_2,\alpha_2'}...\delta_{\alpha_N,\alpha_N'}\end{aligned} \tag{1.42}$$

and completeness given by

$$\sum_{\alpha_1\alpha_2...\alpha_N} |\alpha_1\alpha_2...\alpha_N)(\alpha_1\alpha_2...\alpha_N| = 1. \tag{1.43}$$

Again, these product states do not incorporate the correct symmetry and one must therefore project out the linear combination which is symmetric or antisymmetric. This is accomplished by using the antisymmetrizer for N particle states containing fermions

$$A = \frac{1}{N!}\sum_p (-1)^p P, \tag{1.44}$$

where the sum is over all $N!$ permutations, P is a permutation operator for N particles, and the sign indicates whether the corresponding permutation is even or odd. An even or odd permutation is decided by the whether the corresponding number of two-particle permutation operators that are necessary to accomplish the permutation in question is even or odd. A symmetrizer must be used for N identical bosons

$$S = \frac{1}{N!}\sum_p P. \tag{1.45}$$

Normalized antisymmetrical states are then given by

$$|\alpha_1\alpha_2...\alpha_N\rangle = \sqrt{N!}\; A\; |\alpha_1\alpha_2...\alpha_N) \tag{1.46}$$

while for bosons one obtains

$$|\alpha_1\alpha_2...\alpha_N\rangle = \left[\frac{N!}{n_\alpha! n_{\alpha'}!...}\right]^{1/2} S\; |\alpha_1\alpha_2...\alpha_N) \tag{1.47}$$

with $\sum_\alpha n_\alpha = N$.

A consequence of this explicit construction of antisymmetric states for N fermions is that no sp state can be occupied by two particles, *i.e.* the quantum numbers represented *e.g.* by α_1 cannot occur twice in any antisymmetric N-particle state. Pauli's exclusion principle is therefore incorporated. For a given antisymmetric N-particle state there are $N!$ physically equivalent states obtained by a permutation of the sp quantum numbers. Only one physical state corresponds to these $N!$ states. By using a standard ordering of the sp quantum numbers one can write the completeness relation for N particles as

$$\sum_{\alpha_1\alpha_2...\alpha_N}^{ordered} |\alpha_1\alpha_2...\alpha_N\rangle\langle\alpha_1\alpha_2...\alpha_N| = 1. \tag{1.48}$$

Example: For three fermions for example, one has

$$|\alpha_1\alpha_2\alpha_3\rangle = \frac{1}{\sqrt{6}}\big\{|\alpha_1\alpha_2\alpha_3) - |\alpha_2\alpha_1\alpha_3) + |\alpha_2\alpha_3\alpha_1) - |\alpha_3\alpha_2\alpha_1) + |\alpha_3\alpha_1\alpha_2) - |\alpha_1\alpha_3\alpha_2)\big\}.$$

It should be clear now that antisymmetry upon interchange of any two particles is incorporated since

$$|\alpha_1\alpha_2\alpha_3\rangle = \; -|\alpha_2\alpha_1\alpha_3\rangle \tag{1.49}$$

and so on. For three bosons one has for example

$$\begin{aligned}|\alpha_1\alpha_1\alpha_2\rangle &= \frac{1}{\sqrt{3!2!}}\big\{|\alpha_1\alpha_1\alpha_2) + |\alpha_1\alpha_1\alpha_2) + |\alpha_1\alpha_2\alpha_1) \\ &\quad + |\alpha_2\alpha_1\alpha_1) + |\alpha_2\alpha_1\alpha_1) + |\alpha_1\alpha_2\alpha_1)\big\} \\ &= \frac{1}{\sqrt{3}}\big\{|\alpha_1\alpha_1\alpha_2) + |\alpha_1\alpha_2\alpha_1) + |\alpha_2\alpha_1\alpha_1)\big\}.\end{aligned}$$

Symmetry upon interchange of any two particles is again incorporated since

$$|\alpha_1\alpha_1\alpha_2\rangle = \; +|\alpha_1\alpha_2\alpha_1\rangle \tag{1.50}$$

and so on.

In the case of a 1-dimensional harmonic oscillator this ordering procedure is obvious but in other cases no ambiguity need arise. If no ordering is employed, completeness is given by

$$\frac{1}{N!}\sum_{\alpha_1\alpha_2\ldots\alpha_N} |\alpha_1\alpha_2...\alpha_N\rangle\,\langle\alpha_1\alpha_2...\alpha_N| = \; 1. \tag{1.51}$$

Normalization for states with ordered sp quantum numbers has the form

$$\begin{aligned}\langle\alpha_1\alpha_2...\alpha_N|\alpha_1'\alpha_2'...\alpha_N'\rangle &= \langle\alpha_1|\alpha_1'\rangle\langle\alpha_2|\alpha_2'\rangle...\langle\alpha_N|\alpha_N'\rangle \\ &= \delta_{\alpha_1,\alpha_1'}\delta_{\alpha_2,\alpha_2'}...\delta_{\alpha_N,\alpha_N'},\end{aligned} \tag{1.52}$$

whereas, if the sp states are not ordered, the result is obtained in the form of a determinant

$$\langle\alpha_1\alpha_2...\alpha_N|\alpha_1'\alpha_2'...\alpha_N'\rangle = \begin{vmatrix} \langle\alpha_1|\alpha_1'\rangle & \langle\alpha_1|\alpha_2'\rangle & \dots & \langle\alpha_1|\alpha_N'\rangle \\ \langle\alpha_2|\alpha_1'\rangle & \langle\alpha_2|\alpha_2'\rangle & \dots & \langle\alpha_2|\alpha_N'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\alpha_N|\alpha_1'\rangle & \langle\alpha_N|\alpha_2'\rangle & \dots & \langle\alpha_N|\alpha_N'\rangle \end{vmatrix}. \tag{1.53}$$

The normalized N-particle wave function of an antisymmetric state is given by

$$\psi_{\alpha_1\alpha_2...\alpha_N}(x_1x_2...x_N) = (x_1x_2...x_N|\alpha_1\alpha_2...\alpha_N\rangle, \tag{1.54}$$

where

$$(x_1x_2...x_N| = \langle x_1|\langle x_2|...\langle x_N| \tag{1.55}$$

and $x_1 = \{\boldsymbol{r}_1, m_{s_1}\}$. Often this wave function is written in determinantal form

$$\psi_{\alpha_1\alpha_2...\alpha_N}(x_1x_2...x_N) = \frac{1}{\sqrt{N!}}\begin{vmatrix} \langle x_1|\alpha_1\rangle & \dots & \langle x_N|\alpha_1\rangle \\ \langle x_1|\alpha_2\rangle & \dots & \langle x_N|\alpha_2\rangle \\ \vdots & \ddots & \vdots \\ \langle x_1|\alpha_N\rangle & \dots & \langle x_N|\alpha_N\rangle \end{vmatrix}. \tag{1.56}$$

Such a wave function is commonly called a Slater determinant [Slater (1929)] . Exchange of rows and columns in a Slater determinant does not change their practical use so both conventions are found in the literature. In practice it is very cumbersome to work with Slater determinants and calculate matrix elements of operators between many-particle states. For this reason a more practical method is introduced in the next chapter.

For N boson states there is no restriction on the occupation of a given sp state. In fact, all particles can occupy the same sp state! For a given symmetric N-particle state there are $N!$ physically equivalent states obtained by a permutation of the sp quantum numbers. In addition, one can have multiple occupation of a sp state. Such states should only be counted once in the completeness relation. In an unrestricted sum over quantum numbers for $N = 3$ all states

$$|\alpha_1\alpha_1\alpha_2\rangle = |\alpha_1\alpha_2\alpha_1\rangle = |\alpha_2\alpha_1\alpha_1\rangle \tag{1.57}$$

occur. The appropriate weighting of these states is obtained by including factorial factors $n_\alpha!$ in the completeness relation as follows

$$\sum_{\alpha_1\alpha_2...\alpha_N} \frac{n_{\alpha_1}!n_{\alpha_2}!...}{N!} \left|\alpha_1\alpha_2...\alpha_N\right\rangle \left\langle\alpha_1\alpha_2...\alpha_N\right| = 1. \tag{1.58}$$

When ordering of the sp states is considered no such factors need be included

$$\sum_{\alpha_1\alpha_2...\alpha_N}^{ordered} \left|\alpha_1\alpha_2...\alpha_N\right\rangle \left\langle\alpha_1\alpha_2...\alpha_N\right| = 1 \tag{1.59}$$

as in the case for fermions. Normalization for states with ordered sp quantum numbers has the form

$$\begin{aligned} \langle\alpha_1\alpha_2...\alpha_N|\alpha'_1\alpha'_2...\alpha'_N\rangle &= \langle\alpha_1|\alpha'_1\rangle\langle\alpha_2|\alpha'_2\rangle...\langle\alpha_N|\alpha'_N\rangle \\ &= \delta_{\alpha_1,\alpha'_1}\delta_{\alpha_2,\alpha'_2}...\delta_{\alpha_N,\alpha'_N}, \end{aligned} \tag{1.60}$$

whereas if the sp states are not ordered one has

$$\langle\alpha_1\alpha_2..\alpha_N|\alpha'_1\alpha'_2..\alpha'_N\rangle = \frac{1}{\left[n_{\alpha_1}!..n_{\alpha'_1}!..\right]^{1/2}} \sum_P \langle\alpha_1|\alpha'_{p_1}\rangle\langle\alpha_2|\alpha'_{p_2}\rangle..\langle\alpha_N|\alpha'_{p_N}\rangle. \tag{1.61}$$

The sum on the right-hand side is called a permanent. The normalized N-particle wave function of a symmetric state is also given by

$$\psi_{\alpha_1\alpha_2...\alpha_N}(x_1x_2...x_N) = (x_1x_2...x_N|\alpha_1\alpha_2...\alpha_N\rangle. \tag{1.62}$$

1.6 Exercises

(1) Determine the expectation values of the kinetic energy for N particles in terms of the relevant single-particle matrix elements by using the Slater determinant (1.56).

(2) Suppose that the single-particle Hilbert space has finite dimension D and is spanned by an orthonormal basis set $\{|\alpha\rangle\}$, $\alpha = 1, \ldots, D$. What is the dimension of the N-fermion Fock space? Comment on the result that the same dimension is obtained for $D - N$ fermions.